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CONDITIONAL AND UNCONDITIONAL NONLINEAR STABILITY IN FLUID DYNAMICS

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A Thesis presented for the degree of
Doctor of Philosophy

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Dedicated to my parents

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Abstract

In this thesis we examine some of the interesting aspects of stability for some convection problems. Specifically, the first part of the thesis deals with the Bénard problem for various Non-Newtonian fluids, whereas the second part develops a stability analysis for convection in a porous medium.

The work on stability for viscoelastic fluids includes nonlinear stability analyses for the second grade fluid, the generalised second grade fluid, the fluid of dipolar type and the fluid of third grade. It is worth remarking that throughout the work the viscosity is supposed to be any given function of temperature, with the first derivative bounded above by a positive constant.

The connection between the two parts of the thesis is made through the method used to approach the nonlinear stability analysis, namely the energy method. It is shown in the introductory chapter how this method works and what are its advantages over the linear analysis.

Nonlinear stability results established in both Part I and Part II are the best one can get for the considered physical situations. Different choices of energy have been considered in order to achieve conditional or unconditional nonlinear stability results.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, the Department of Mathematical Science, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Summary

This thesis examines a variety of stability results for various convection problems. Among the numerous real phenomena which involve convection, there are two on which we concentrate here: convection in a layer of a fluid heated below and convection in a porous medium. Both parts of this thesis analyse the stability for some of these situations.

To begin, we introduce the energy method, which we shall employ throughout the thesis in order to derive nonlinear stability criteria. We illustrate this method within a complete study of the stability for an incompressible and homogeneous Navier-Stokes fluid, when the viscosity is considered constant. We examine the stability and instability boundaries using linear theory, and then by the energy method.

In Part I we study the effect of variation of the temperature-dependent viscosity on the flow of Non-Newtonian fluids heated from below. The generalised fluids on which we focus our analysis here are: the fluid of second grade, the fluid of generalised second grade, the dipolar fluid and the fluid of third grade. A complete nonlinear stability analysis is delivered for each of the fluids and we show that the analysis and the results are different with respect to the presence of some nonlinear terms in the stress tensor of each particular fluid. It is of interest to mention that the strongest result is achieved in the stability analysis for the third grade fluid, due to the extra structure provided by the constitutive equations.

The object of Part II is to investigate the stability of penetrative convection in a porous layer, beneath the ocean bed and above the interface of the thawing subsea permafrost ground. In Part I, of most interest were the changes in viscosity, whereas here we are mostly concerned about variations of the temperature and the

water salinity and their effects on the stability. A linearised analysis is performed first to deliver a linear instability boundary. Nonlinear energy results turn out to be very close to those of the linear analysis, eliminating the gap between the instability and stability boundaries. Two energy analyses are presented in order to improve the nonlinear stability criterion. A generalised energy leads to a conditional nonlinear result, whereas an weighted energy is necessary for a stronger mathematical result, an unconditional one, i.e. with no restrictions on the initial amplitudes.

Some conclusions and speculations are presented at the end of this thesis. It is worth remarking at this point that all the results from Part I and Part II are new and most of the work has been already published or submitted for publication.

Chapter 1

Introduction and background

1.1 Notations

The notation used throughout the thesis is standard, with indicial and bold face notation for vectors and tensors.

The symbol Δ is the three-dimensional Laplacian operator and Δ^* denotes the Laplacian operator with respect to the x_1, x_2 variables, i.e.

$$\Delta \mathbf{u} \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2},$$
$$\Delta^* \mathbf{u} \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

Standard indicial notation with the Einstein summation convection is employed

$$u_{i,i} \equiv div\, \mathbf{u} \equiv \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3},$$
$$u_j u_{i,j} \equiv (\mathbf{u} \cdot \nabla)\, \mathbf{u} \equiv u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}.$$

A subscript t denotes partial differentiation with respect to the time, whereas a superposed dot is used for the material derivative, e.g.,

$$\dot{u}_i \equiv \frac{\partial u_i}{\partial t} + u_j u_{i,j} = u_{i,t} + u_j u_{i,j}.$$

As most of the applications deal with motion in a vertical plane, $z \in (0, d)$, we regard all the functions as periodic in x and y. We shall denote by V a periodic cell defined by the Cartesian product of a repetitive plan-form in (x, y) plan with $(0, d)$, whereas Γ denotes the boundary of the periodic cell V lying in the plane $z = d$. $L_2(V)$ is the space of square-integrable functions on V and $W^{1,2}(V)$ is the space of functions $f \in L^2(V)$ with the first derivative in $L^2(V)$, as well; $\langle . \rangle$ denotes the integration over V and $||.||$ is the $L_2(V)$ norm, e.g.,

$$\int_V f\, g\, dV = \langle f\, g \rangle,$$
$$\int_V f^2\, dV = ||f||^2.$$

$D(.)$ is the Dirichlet integral over V, i.e.

$$D(f) = ||\nabla f||^2 = \int_V f_{,i}\, f_{,i}\, dV.$$

1.2 Stability theory

The concept of stability in the mathematical study of a physical system has had a long and fruitful history. Real situations show that for the practical use of many technical systems stability properties can be a decisive criterion. Some examples where stability properties are important could be: engineering structures (bridges, plates, shells structures under pressure loading or unloading by flowing fluids), vehicles moving at high speed, truck-trailer combinations, railway trains, hydrodynamics problems.

Over the past decades, engineers have approached many of their stability problems using a linearised stability analysis. In addition, if a linear stability analysis does not seem to be sufficient, numerical simulations are employed. Such a numerical simulation allows one to check whether a linearised analysis provides practically useful results or not. However, contrary to the widespread belief that linearised stability analysis together with numerical simulation are a general method of treating stability problems, it has been proved that this is not the case. There exists a large number of problems where a linearised analysis does not give much information about the behaviour of the nonlinear system at all and, hence, a numerical simulation would be very costly without yielding much insight into the qualitative behaviour. Alternatively, a wide large range of nonlinear problems can be analysed and solved in a straightforward manner making use only of an appropriate mathematical background.

We shall roughly explain in this introductory part what we mean by a nonlinear investigation, when it is absolutely necessary to perform a nonlinear analysis, whether or not there are restrictions in carrying out a nonlinear stability analysis and what is the advantage of a nonlinear analysis over a linear one.

To illustrate some of the mechanisms and concepts of stability we shall now work through a classical problem.

Bénard convection. Thermal instability often arises when a fluid is heated from below. The classical example, described in this section, is a horizontal layer of an incompressible and homogeneous Navier-Stokes fluid with its lower side hotter than its upper. The basic state is then one of rest with light fluid below heavy fluid. When the temperature difference across the layer is great enough, the stabilising effects of viscosity and thermal conductivity are overcome by the destabilising buoyancy, and an overturning instability ensues thermal convection. The convection in a horizontal layer of fluid heated from below is called *Bénard convection.*

Consider then a layer of incompressible viscous fluid occupying the infinite horizontal layer contained between the planes $z = 0$ and $z = d$, with velocity vector $\mathbf{v}$ and pressure p. The partial differential equations governing this situation are

$$\begin{aligned} \dot{v}_i &= -\frac{1}{\rho}p_{,i} + \nu\,\Delta v_i - k_i g[1 - \alpha(T - T_0)], \\ v_{i,i} &= 0, \\ \dot{T} &= \kappa \Delta T, \end{aligned} \tag{1.1}$$

where ρ, ν, g, α, T, κ are respectively, density, kinematic viscosity, gravity, thermal expansion coefficient, temperature, thermal diffusivity and $\mathbf{k}$ is the vector $(0, 0, 1)$. We treat here $\rho\,(= \rho_0)$ and $\nu\,(= \nu_0)$ as constants. We remark that we have employed the Navier-Stokes equations with a Boussinesq approximation; the basis for this approximation is that there are flows in which the temperature varies little, and therefore the density varies little, yet in which the buoyancy drives the motion. Then, the variation of density is neglected everywhere except in the buoyancy.

From the no slip in velocity condition, we write

$$v_i = 0, \quad z = 0, d. \tag{1.2}$$

If we consider that the temperatures at the boundaries are fixed,

$$T = T_0, \quad z = 0 \quad \text{and} \quad T = T_1, \quad z = d, \tag{1.3}$$

where $T_0 > T_1$ (i.e. the fluid is heated from below), then the boundary value problem (1.1)-(1.3) has a steady solution $(\bar{\mathbf{v}}, \bar{T}, \bar{p})$

$$\bar{\mathbf{v}} \equiv 0, \quad \bar{T} = -\beta\, z + T_0,$$

with β being a measure of the temperature gradient

$$\beta = \frac{T_0 - T_1}{d} > 0.$$

The pressure $\bar{p}$ is determined from $(1.1)_1$, up to a constant, namely

$$\bar{p} = \rho_0\, g\, z + \frac{\rho_0 g \alpha \beta}{2} z^2.$$

A real system is always subject to some small fluctuations which may provide perturbations from the stationary solution. If these perturbations are amplified with time, then the evolution naturally drives the system away from that stationary state. If, however, perturbations from the stationary solution decay in time, then the steady solution it is called stable. From a practical point of view it is then necessary that all disturbances decay rapidly.

An important task of stability theory is to separate the stable solutions from the unstable ones, i.e. to generate a stability boundary. In order to test stability or instability of the stationary solution, let $(\mathbf{u}, \theta, \pi)$ denote the perturbation from the steady solution. The total flow $(\bar{\mathbf{v}} + \mathbf{u})$ must satisfy the equations of motions and the same boundary conditions as $\mathbf{v}$, but the perturbation flow is otherwise arbitrary, in particular it is not necessarily small. One wishes to know if and under which conditions the two solutions *come together* or *stay apart.* To answer this question we form the initial-boundary problem governing the evolution of the disturbances

$$\begin{aligned} \dot{u_i} &= -\frac{1}{\rho_0}\, \pi_{,i} + \nu_0\, \Delta u_i + k_i\, g\, \alpha\, \theta, \\ u_{i,i} &= 0, \\ \dot{\theta} &= \kappa \Delta \theta + \beta w, \end{aligned} \tag{1.4}$$

where $w = u_3$, along with the appropriate boundary conditions

$$\theta = u_i = 0, \quad z = 0, d.$$

We now introduce the dimensionless variables (the star ones) and consider the following notations

$$\begin{array}{c} x_i = x_i^* d, \quad u_i = u_i^* U, \quad \pi = \pi^* P, \quad \theta = \theta^* \tilde{T}, \quad t = t^* \mathcal{T}, \\ U = \dfrac{\nu_0}{d}, \quad P = \dfrac{U \rho_0 \nu_o}{d}, \quad \tilde{T} = U \sqrt{\dfrac{Pr\, \beta}{\alpha g}} \\ \mathcal{T} = \dfrac{d^2}{\nu_0}, \quad Pr = \dfrac{\nu_0}{\kappa}, \quad R = \sqrt{\dfrac{\alpha \beta g d^4}{\nu_0 \kappa}}, \end{array}$$

where Pr is the Prandtl number and $Ra = R^2$ is the Rayleigh number. The non-dimensionalized system is then (dropping all the stars)

$$\begin{aligned} \dot{u}_i &= -\pi_{,i} + \Delta u_i + k_i\, R\, \theta, \\ u_{i,i} &= 0, \\ Pr\, \dot{\theta} &= \Delta \theta + R\, w, \end{aligned} \tag{1.5}$$

and the corresponding boundary conditions are

$$\theta = u_i = 0, \quad z = 0, 1. \tag{1.6}$$

Note that the Rayleigh number is positive when the lower boundary is the hotter one ($T_0 > T_1$) and it is a characteristic ratio of the destabilising effect of buoyancy to the stabilising effects of diffusion and dissipation. Also note that the Prandtl number is an intrinsic property of the fluid, not of the flow, it measures the ratio of materials parameters.

The aim of stability theory is to extract information about the evolution of the perturbations without solving the full nonlinear boundary problem in (1.5)-(1.6).

Linear stability analysis. We start the stability study of the steady solution of (1.1) with a linearised analysis. Therefore, we neglect the nonlinear terms in (1.5) and the system to be solved is now

$$\begin{aligned} u_{i,t} &= -\pi_{,i} + \Delta u_i + k_i\, R\, \theta, \\ u_{i,i} &= 0, \\ Pr\, \theta_{,t} &= \Delta\theta + R\, w, \end{aligned} \tag{1.7}$$

where the boundary conditions (1.6) are considered. Since (1.7) and (1.6) are linear, the time evolution of this system of equations may be reduced to an eigenvalue problem by imposing a time dependence for solutions, i.e.,

$$u_i(\mathbf{x},t) = e^{\sigma t} u_i(\mathbf{x}), \quad \theta(\mathbf{x},t) = e^{\sigma t}\theta(\mathbf{x}), \quad \pi(\mathbf{x},t) = e^{\sigma t}\pi(\mathbf{x}), \tag{1.8}$$

with σ being a complex constant. Next from (1.7) we may derive

$$\begin{aligned} \sigma\, u_i &= -\,\pi_{,i} + \Delta u_i + k_i R\theta, \\ u_{i,i} &= 0, \\ \sigma\, Pr\theta &= \Delta\theta + Rw. \end{aligned} \tag{1.9}$$

The operator whose spectrum is of interest in (1.9) is not generally self-adjoint, so the eigenvalues may be complex numbers.

We shall show that in the particular example we discuss here, the rate of growth, σ, is real. If we assume $\sigma = \sigma_r + i\,\sigma_i$, σ_r, $\sigma_i \in \mathbb{R}$, then we expect to have complex solutions u_i and θ, with the associated complex conjugates u_i^*, respectively θ^*. We now multiply $(1.9)_1$ by u_i^*, $(1.9)_3$ by θ^*, and integrate over the periodic cell V to obtain

$$\begin{aligned} \sigma\,\langle u_i\, u_i^*\rangle &= R\langle\theta\, w^*\rangle - \langle u_{i,j}\, u_{i,j}^*\rangle, \\ Pr\,\sigma\,\langle\theta\,\theta^*\rangle &= -\langle\theta_{,i}\,\theta_{,i}^*\rangle + R\langle w\,\theta^*\rangle. \end{aligned}$$

Adding the last two identities it follows that

$$\sigma\,[||\mathbf{u}||^2 + Pr\,||\theta||^2] = R[\langle\theta\, w^*\rangle + \langle w\,\theta^*\rangle] - \langle u_{i,j}\, u_{i,j}^*\rangle - \langle\theta_{,i}\,\theta_{,i}^*\rangle. \tag{1.10}$$

Since

$$\begin{aligned}\langle \theta\, w^* \rangle + \langle w\, \theta^* \rangle &= \langle (\theta_r + i\theta_i)\,(w_r - iw_i) \rangle + \langle (w_r + iw_i)\,(\theta_r - i\theta_i) \rangle \\ &= 2\,[\langle \theta_r\, w_r \rangle + \langle w_i\, \theta_i \rangle] \in \mathbb{R},\end{aligned}$$

one can see that the imaginary part of (1.10) is

$$\sigma_i\,[||\mathbf{u}||^2 + Pr\,||\theta||^2] = 0.$$

Therefore, $\sigma_i = 0$ and the proof of $\sigma \in \mathbb{R}$ is completed.

As σ is a real number, the linearised equations (1.7) satisfy the *Principle of exchange of stability.* When this principle holds, convection set in as *stationary convection.* Now recalling (1.8), if $\sigma > 0$ for all modes, the corresponding disturbance will be amplified, growing exponentially in time until it is so large that nonlinearities become significant and the mode is called *unstable.* In the case when the largest eigenvalue is zero, the mode is said to be marginal or *critically stable*, whereas if $\sigma < 0$ such that the disturbances decay exponentially it is said that the mode is *asymptotically stable.*

A small disturbance of the basic flow will in general excite all modes, so that if $\sigma > 0$ for at least one mode, then the flow is unstable. Conversely, if $\sigma \leq 0$ for all of the complete sets of modes, then the flow is stable. In conclusion, it is very important for one to know the characteristics of the critical flow, precisely when $\sigma = 0$.

We shall then solve the linearised problem (1.9) for $\sigma = 0$, namely

$$\begin{aligned}k_i\, R\theta + \Delta u_i &= \pi_{,i}, \\ u_{i,i} &= 0, \\ \Delta\theta + Rw &= 0.\end{aligned} \tag{1.11}$$

In order to reduce system (1.11) to a one-dimensional eigenvalue problem, we first take the third component of the operation curl curl of equation $(1.11)_1$ and using $(1.11)_2$ we obtain

$$-R\,\Delta^*\theta - \Delta^2 w = 0, \tag{1.12}$$

We now adopt a normal mode representation of form $\theta(x,y,z) = \Theta(z)h(x,y)$, $w(x,y,z) = W(z)h(x,y)$, with $\Theta(z)$ and $W(z)$ being z-dependent functions and

$h(x, y)$ is a planform which tiles the plane (x, y) and satisfies the equation (see eg. Christoperson [9])

$$\Delta^* h = -k^2 h, \tag{1.13}$$

where k^2 is an arbitrary constant arising from the separation of the variables. The equation (1.13) is well known, called *the reduced wave equation* (*Helmholtz equation, membrane equation*), thus k is interpreted as a real horizontal wavenumber. In general, a disturbance excites components for each value of k.

Considering $D = d/dz$, $z \in (0, 1)$, $(x, y) \in \mathbb{R}^2$ and replacing the normal mode representations of θ and w in $(1.11)_3$ and (1.12), the system is reduced to

$$\begin{aligned} Rk^2\Theta - (D^2 - k^2)^2 W &= 0, \\ RW + (D^2 - k^2)\Theta &= 0, \end{aligned} \tag{1.14}$$

where the boundary conditions are given by

$$W = DW = \Theta = 0, \quad z = 0, 1. \tag{1.15}$$

It may be observed from the equations (1.14), that in fact Pr does not affect the conditions for the critical stability.

The eigenvalue problem (1.14)-(1.15) is solved by the compound matrix method (see Appendix B) and provides a critical Rayleigh number of the linear analysis, which we denote by Ra_L. Hence, the critical linear Rayleigh number is that value Ra_L for which $\sigma(k, Ra) > 0$ for some k, whenever $Ra > Ra_L$, and $\sigma(k, Ra) \leq 0$ for all k, whenever $Ra \leq Ra_L$. In conclusion, for given critical values of the parameters on which the eigenvalue σ depends, i.e., Ra and k, the flow is unstable if $\sigma > 0$ for any mode with any real value of k, and stable is $\sigma \leq 0$ for all modes. Therefore, the linearization determines the parameter value for which loss of stability occurs, a value which can be calculated. Precisely, linear instability analysis provides a sufficient condition for instability.

This leads to the important conclusion that whether a linear analysis is sufficient to describe the stability behaviour of a structure or not, can only be determined from a nonlinear analysis!

Nonlinear stability analysis. To obtain sufficient conditions for stability with respect to arbitrary disturbances the full nonlinear equations must be considered. In order to establish the nonlinear stability of the steady solution, it is sufficient to show that all perturbations vanish rapidly as $t \longrightarrow \infty$. For this is sufficient to prove that any relevant perturbation vanishes exponentially. One suitable way to demonstrate this is *the energy method*. A fuller account of the energy method and its applications on a various problems may be found in Straughan [64].

We return to the system (1.5) with the boundary conditions (1.6) and form the energy identities multiplying $(1.5)_1$ by u_i, $(1.5)_3$ by θ and integrating over the period cell V. After use of integration by parts and the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}\,||\mathbf{u}||^2 = R\,\langle\theta\, w\rangle - ||\nabla\mathbf{u}||^2, \tag{1.16}$$

$$\frac{1}{2}\frac{d}{dt}\,Pr\,||\theta||^2 = R\,\langle\theta\, w\rangle - ||\nabla\theta||^2. \tag{1.17}$$

We now add equation (1.16) and equation (1.17) multiplied by λ, a positive coupling parameter to be selected later, to develop a variational problem. The resulting energy identity is

$$\frac{1}{2}\frac{d}{dt}\,||\mathbf{u}||^2 + \lambda\frac{1}{2}\frac{d}{dt}\,Pr\,||\theta||^2 = R\,(1+\lambda)\,\langle\theta\, w\rangle - ||\nabla\mathbf{u}||^2 - \lambda\,||\nabla\theta||^2. \tag{1.18}$$

The next step is to define an *energy* by

$$E(t) = \frac{1}{2}\,||\mathbf{u}||^2 + \frac{1}{2}\lambda\,Pr\,||\theta||^2.$$

We do stress that the energy considered here has a *natural* form, as one may observe from equations (1.16)-(1.17). If we set now

$$\begin{aligned} \mathcal{I} &= (1+\lambda)\langle\theta\, w\rangle, \\ \mathcal{D} &= ||\nabla\mathbf{u}||^2 + \lambda||\nabla\theta||^2, \end{aligned}$$

from (1.18) immediately follows

$$\frac{dE}{dt} = R\mathcal{I} - \mathcal{D}.$$

We derive further

$$\frac{dE}{dt} = -R\mathcal{D}\left(\frac{1}{R} - \frac{\mathcal{I}}{\mathcal{D}}\right) \le -R\mathcal{D}\left(\frac{1}{R} - \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}\right), \tag{1.19}$$

where $\mathcal{H}$ is the space of admissible functions over which the maximum is sought,

$$\begin{aligned}\mathcal{H} &= \{u_i, \theta \,|\, u_i \in W^{1,2}(V), u_{i,i} = 0, u_i = 0 \ \text{ at } z = 0, 1; \\ &\quad \theta \in W^{1,2}(V), \theta = 0 \ \text{ at } z = 0, 1\}\end{aligned}$$

with u_i and θ satisfying a plane tiling periodic planform in x and y. Let us define

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}}, \tag{1.20}$$

then recalling this definition in (1.19), it turns out that

$$\frac{dE}{dt} \leq -R\mathcal{D}(\frac{1}{R} - \frac{1}{R_E}). \tag{1.21}$$

One may observe that provided

$$R < R_E, \tag{1.22}$$

so $R^{-1} - {R_E}^{-1} = d > 0$, then from (1.21) it follows

$$\frac{dE}{dt} \leq -d\, R\, \mathcal{D}.$$

It should be noted here that due to the definitions of $\mathcal{D}$ and E, there exist a positive constant ψ, by Poincaré's inequality, such that $\mathcal{D} \geq \psi\, E$. This result is crucial in proving the nonlinear stability result. Hence

$$\frac{dE}{dt} \leq -d\, \psi\, R\, E,$$

which by integration leads to

$$E(t) \leq e^{-d\psi Rt} E(0). \tag{1.24}$$

From (1.24) we deduce that all disturbances decay very rapidly. Therefore, we have established that provided $R < R_E$, then the stationary solution of (1.1) is stable and moreover asymptotically stable.

The criterion of importance is then (1.22), and everything is reduced to solving the maximum problem (1.20).

The Euler Lagrange equations for the maximum problem (1.20) are derived as follows

$$\begin{aligned}\frac{d}{d\epsilon} \frac{\mathcal{I}(\theta + \epsilon\, t, u_i + \epsilon\, \eta_i)}{\mathcal{D}(\theta + \epsilon\, t, u_i + \epsilon\, \eta_i)}\Big|_{\epsilon=0} &= \delta(\frac{\mathcal{I}}{\mathcal{D}}) = \frac{\mathcal{D}\, \delta\mathcal{I} - \mathcal{I}\, \delta\mathcal{D}}{\mathcal{D}^2} \\ &= \frac{1}{\mathcal{D}} [\delta\mathcal{I} - \frac{\mathcal{I}}{\mathcal{D}}\Big|_{\max_{\mathcal{H}}} \delta\mathcal{D}] \\ &= \frac{1}{\mathcal{D}} [\delta\mathcal{I} - \frac{1}{R_E} \delta\mathcal{D}] = 0,\end{aligned}$$

where t, η_i are arbitrary $C^2(0,1)$ functions with $t(0) = t(1) = 0$, $\eta_i(0) = \eta_i(1) = 0$.

Hence,

$$\delta\mathcal{I} - \frac{1}{R_E}\,\delta\mathcal{D} = 0, \tag{1.25}$$

with

$$\begin{aligned} \delta\mathcal{I} &= \frac{d}{d\epsilon}\int_V (1+\lambda)\,(\theta+\epsilon\, t)\,(w+\epsilon\,\eta_3)\,dV|_{\epsilon=0}, \\ \delta\mathcal{D} &= \frac{d}{d\epsilon}\int_V [\,\lambda\,|\theta_{,i}+\epsilon\, t_{,i}|^2 \,+\, |u_{i,j}+\epsilon\,\eta_{i,j}|^2]\,dV|_{\epsilon=0}. \end{aligned}$$

Upon integrating by parts, (1.25) leads to

$$\int_V (1+\lambda)\,(t\,w+\theta\,\eta_3)\,dV \,+\, \frac{1}{R_E}\int_V (2\lambda\,\Delta\theta\, t + 2\Delta u_i\eta_i)\,dV = 0,$$

or, further

$$\int_V [(1+\lambda)w \,+\, \frac{1}{R_E}\,2\lambda\,\Delta\theta]\,t\,dV \,+\, \int_V [(1+\lambda)\theta\, k_i \,+\, \frac{1}{R_E}\,2\,\Delta u_i]\,\eta_i\,dV = 0.$$

Since t and η_i are arbitrary functions, we must have

$$\begin{aligned} (1+\lambda)w \,+\, \frac{1}{R_E}\,2\lambda\,\Delta\theta &= 0, \\ (1+\lambda)\theta\,k_i \,+\, \frac{1}{R_E}\,2\,\Delta u_i &= 0. \end{aligned}$$

Since $\mathcal{H}$ is restricted to those functions that are divergence free, we must add into the maximum problem the constraint $u_{i,i} = 0$ multiplied by a Lagrange multiplier, $p(x)$, $(\int_V p(x)u_{i,i}dx = 0)$.

Therefore, the Euler Lagrange equations for the maximum problem (1.20), which give an eigenvalue problem for R_E, are

$$\begin{aligned} (1+\lambda)R_E\, w + 2\lambda\,\Delta\theta &= 0, \\ (1+\lambda)R_E\,\theta\,k_i + 2\,\Delta u_i &= 2\,p_{,i}. \end{aligned} \tag{1.26}$$

We now take the third component of (curl curl)$(1.26)_2$ and decompose into normal modes and equations (1.26) become

$$\begin{aligned} (1+\lambda)R_E\, W + 2\,\lambda\,(D^2-k^2)\Theta &= 0, \\ (1+\lambda)R_E\,\Theta\, k^2 - 2\,(D^2-k^2)^2 W &= 0 \end{aligned} \tag{1.27}$$

System (1.27) is solved subject to the boundary conditions

$$W = DW = \Theta = 0, \quad z = 0, 1.$$

This eigenvalue problem is solved numerically by the compound matrix method, with the optimal Rayleigh number of global stability, Ra_E, found by choosing λ such that

$$Ra_E = R_E^2 = \max_{\lambda} \min_{k} R^2(\lambda, k).$$

It turns out from the numerical code, that the optimal value for the coupling parameter λ is 1. We note that if we take $\lambda = 1$ in equations (1.27), one obtain the same equations as those of the linearised analysis, (1.14). Thus the critical Rayleigh number of nonlinear theory that guarantees *stability* is the same as the critical Rayleigh number of linear theory, that yields *instability*. The instability boundary is then the same as the nonlinear one, therefore no subcritical instability may arise.

Finally, we have a complete picture of the stability for the problem which we have considered here. The main idea of using an energy method for nonlinear analysis consists of finding a suitable form of the energy function such that the nonlinear critical Rayleigh number is close to the one from the linearised analysis. The determination of an energy for specific problems is not so obvious as in the classical example considered here. There are situations when a natural energy may deliver a global nonlinear stability criterion, as (1.22), but there are situations when nonlinearities in the governing equations prevent useful analyses, even when more complicated forms of energy are employed. These are the cases when one may achieve nonlinear stability conditional upon a threshold for the initial amplitudes, a so-called *conditional nonlinear stability* result.

PART I. Viscoelastic fluids

Scope and plan of this part. We shall see in the following studies of different fluids of grade n, $n = 2, 3$, how the extra nonlinearities in the stress tensor may change the energy stability analysis and how they can influence the accuracy of the final nonlinear stability results.

Chapter 2 is an introduction to the viscoelastic fluids, with emphasis on the associated constitutive equations. All of the stability analyses are undertaken for a temperature dependent viscosity, as it is stated in one of the sections here.

We next consider in Chapter 3 the second grade fluid in the Bénard problem, and prove that a generalised energy is necessary in order to establish a conditional nonlinear stability criterion. The extra nonlinearities in the stress tensor associated to the second grade fluid, improve the natural energy form, with an additional $||\nabla \mathbf{u}||^2$ term, and the nonlinear analysis is less complicated than in the Navier-Stokes theory. Still the nonlinear stability result relies upon a threshold for the initial amplitudes. A similar procedure works for the generalised second grade fluid, analysis presented in Chapter 4, though a trick is used to carry out a generalised energy analysis. Chapter 5 is devoted to the study of stability of another non-Newtonian viscous fluid, namely the dipolar fluid. One step forward is made here, due to nonlinearities arising in the stress tensor for the dipolar fluid. A natural energy analysis is proved to be sufficient to provide a nonlinear stability criteria, though the result is a conditional one, in the sense that the size of the initial energy amplitude is restricted.

In Chapter 6 we establish that it is the fluid of third grade, where not only is a natural energy strong enough to complete the nonlinear analysis, but the nonlinear stability boundary is found for all initial data, no matter how large they are.

Chapter 2

On viscoelastic fluids

2.1 Viscoelastic fluids

In the introductory chapter of this thesis we have presented one of the best known thermally-induced stability situations, the Bénard problem for a incompressible homogeneous fluid. The constitutive equations associated were taken to be the Navier-Stokes equations with a Boussinesq approximation. The object of the following chapters is to deliver stability results for the Bénard problem in a fluid being more complex than a Newtonian fluid (like air or water). The flow of such fluids (honey, molten plastic, petroleum oil, blood, paints, some greases, etc.) requires effects which are not present in a flow of a Newtonian fluid. The Navier-Stokes theory of incompressible fluids has succeeded in describing the behaviour of certain real fluids, but there are failures in modelling the responses of others, especially of those fluids with a high viscosity.

Thus, many new theories have been proposed in an attempt to study the *non-Newtonian* fluid behaviour. The aim of the non-Newtonian theories is to represent and predict more accurately the behaviour of a narrow class of natural fluids and we expect any of these theories to apply properly to a large variety of natural fluids, as do the classical theories. In order to study these special fluids, one should specify the type of material of which the fluids are made. This, in general, means proposing the constitutive equations of the considered fluid, which relate the stress tensor and heat flux vector to the motion. The stress tensor, which cumulates the contact forces acting on the fluid, is determined - for a simple fluid - by the history of the motion of that fluid.

The relevant model for an incompressible homogeneous fluid, with a Boussinesq approximation, consists in the momentum equation

$$\rho \dot{v}_i = -\rho_0 g \delta_{i3} \left[1 - \alpha(T - T_0)\right] + T_{ji,j}, \tag{2.1}$$

the continuity equation,

$$v_{i,i} = 0, \tag{2.2}$$

and the equation for temperature evolution, reduced to

$$\dot{T} = \kappa \Delta T, \tag{2.3}$$

where v_i, ρ, g, T are, respectively, the velocity, density, gravity and temperature; $\mathbf{T}$ is the stress tensor, T_0 a reference temperature, α the coefficient of thermal expansion and κ the coefficient of thermal diffusivity. The superposed dot denotes the material derivative.

For an incompressible simple fluid, in which case the present stress is determined by the history of the gradients of the deformation, the general constitutive equations for the extra stress tensor, $\mathbf{T}_E = \mathbf{T} + p\,\mathbf{I}$, are

$$\mathbf{T} + p\,\mathbf{I} = \mathcal{F}_{s=0}^{\infty}\left(g(s)\right), \tag{2.4}$$

where p contains all terms which are scalar multiples of the unit tensor $\mathbf{I}$, and $\mathcal{F}$ is an isotropic functional with $\mathcal{F}_{s=0}^{\infty}(0) = 0$. The function $g(s)$ is called the *history* at time s, defined by

$$g(s) = \mathbf{C}_t(t - s) - \mathbf{I}$$

with $\mathbf{C}_t(\tau)$ being the right Cauchy-Green tensor at time τ, relative to the configuration at time t

$$\mathbf{C}_t(\tau) = D_t^T(\tau) D_t(\tau).$$

Here, $D_t(\tau)$ is the gradient of the displacement function $x_i(X, \tau)$, which gives the position of the fluid at time τ having the position X at time t, i.e. $D_t(\tau) = \frac{\partial x_i}{\partial X}$, and the superposed T here denotes the transpose.

As the value of $\mathcal{F}$ vanishes when $g(s) = 0$ (i.e., the fluid has always been at rest), we may conclude that $\mathcal{F}$ describes the response of the fluid to disturbance from the equilibrium state. This functional is called the *response functional.* The response functional does not depend on the reference configuration, as a simple fluid has no preferred configuration and has no permanent memory for any particular state.

The mechanical response of viscoelastic fluids to forces is much more complicated than the response of Newtonian fluids. To study the stability of an initial value problem for a viscoelastic fluid, it is necessary to give an explicit formula for the functional expansion of $\mathcal{F}$. The difficulty of formulating a stability theory for viscoelastic flows is the complexity of response.

First attempts were made by Reiner (1945) and Rivlin (1948); they have proposed an extension for the Navier-Stokes equations for a viscous fluid, considering that the stress components at a point in the fluid depend only on the velocity gradients at that point. Later, Rivlin & Ericksen [56], determined the expressions for the stress components depending on the gradients of displacement, velocity, acceleration, second acceleration, ..., $(n-1)$th acceleration. One class from those many constitutive assumptions that have been employed to study non-Newtonian fluids, a class that gained support from both experimenters and theorists, is that of the *differential type of complexity* n, introduced by the work of Rivlin & Ericksen [56]. For an incompressible fluid, this class is characterised by the stress constitutive assumption

$$\mathbf{T} + p\,\mathbf{I} = f(\mathbf{A}_1, \ldots, \mathbf{A}_n), \tag{2.5}$$

where $\mathbf{A}_1$,...,$\mathbf{A}_n$ are the first n Rivlin-Ericksen tensors given by the recursive relations

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{L} + \mathbf{L}^T, \\ \mathbf{A}_n &= \dot{\mathbf{A}}_{n-1} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1}, \end{aligned}$$

with $\mathbf{L}$ being the spatial gradient of velocity. Any fluid which is modelled by (2.5) is called a Rivlin-Ericksen fluid of complexity n.

Coleman & Noll [11] have applied to constitutive equations of continuum mechanics, an approximation theorem which permits the asymptotic approximation of a memory functional for slow histories by a polynomial function of the derivatives at an initial time of the argument functions of the functional.

What they have defined by a memory functional is $\mathcal{F}$ with a fading memory, i.e., the value of $\mathbf{T} + p\mathbf{I}$ was considered to be more sensitive to the values of g for small s (recent past), than for very large s (distant past). On this account they have introduced a norm on the space of the histories functions, g,

$$||g|| = \begin{cases} \left(\int_0^\infty (|g(t)|\, h(t))^p \, dt \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{t \geq 0} |g(t)|\, h(t), & p = \infty, \end{cases}$$

such that greater influence was assigned to the recent past than to the distant past, by using a real-valued non-negative function $h(t)$, called the *influence function*, whose values vanish as $t \longrightarrow \infty$. This function characterises the rapidity with which the memory is fading.

They also considered a retardation on the history, Γ_α, replacing the given history by one which is essentially the same, but slower

$$(\Gamma_\alpha\, g)(s) = g_\alpha(s),$$

where $\alpha \in (0, 1]$ is the retardation factor.

With these assumptions, they have shown that the response functional is given by

$$\mathcal{F}_{s=0}^{\infty}\left(g_\alpha(s) \right) = \sum_{i=1}^{\infty} S_i\,(\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2, \ldots, \hat{\mathbf{A}}_i), \tag{2.6}$$

where $\hat{\mathbf{A}}_i = \alpha^i\, \mathbf{A}_i$ and $\mathbf{A}_i$ are the Rivlin-Ericksen tensors.

For sufficiently retarded motions, the partial sums

$$\mathbf{S}_{[n]} = \sum_{i=1}^{n} S_i\,(\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2, \ldots, \hat{\mathbf{A}}_i) = \sum_{i=1}^{n} \alpha^i\, S_i\,(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_i),$$

approximate the extra stress tensor, $\mathbf{T} + p\mathbf{I}$, of an incompressible simple fluid with fading memory, with an error of order $n+1$. They have also shown that the Navier-Stokes form of the stress arises as the first order truncation of such a retardation expansion.

For $i = 1, 2, 3, 4$, S_i can be written explicitly, in terms of the Rivlin-Ericksen tensors, like

$$
\begin{aligned}
\mathbf{S}_1 &= \mu\,\mathbf{A}_1,\\
\mathbf{S}_2 &= \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,\\
\mathbf{S}_3 &= \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\,\mathbf{A}_2)\,\mathbf{A}_1,\\
\mathbf{S}_4 &= \gamma_1\,\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2)\\
&\quad + \gamma_5(tr\,\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(tr\,\mathbf{A}_2)\,\mathbf{A}_1^2 + [\gamma_7\,tr\mathbf{A}_3 + \gamma_8\,tr(\mathbf{A}_2\mathbf{A}_1)]\,\mathbf{A}_1,
\end{aligned}
$$

with μ, α_1, α_2, β_1, β_2, β_3, $\gamma_1, \gamma_2, \dots \gamma_8$ constants or temperature-dependent material moduli. They have introduced the notion of *fluid of grade n* being that fluid of which the stress tensor takes the following form

$$
\mathbf{T} = -p\,\mathbf{I} + \mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_n, \tag{2.7}
$$

with $\mathbf{S}_1$, $\mathbf{S}_2$, $\dots, \mathbf{S}_n$ defined as above. As one can easily observe, the fluids of grade n form a subclass of the differential type of complexity n fluids, the one for which the functional f of (2.5) is a polynomial function. The fluids of grade n are viewed as approximations for certain non-Newtonian fluids. However, the constitutive assumptions from above may also be considered as exact models for some fluids. In other words, we may regard (2.7) as being an exact equation modelling the behaviour of a class of materials in all motions. I emphasise here that is not the object of this research work to decide whether or not one particular model of non-Newtonian fluids is able to capture all the physics of some particular natural material. A detailed discussion on this matter can be found in the literature; see Truesdell & Noll [71], Green & Laws [29], Dunn & Fosdick [13], Joseph [40], Fosdick & Rajagopal [19]- [20], Rajagopal [52], Dunn & Rajagopal [14] for a comprehensive review of this subject.

In order to proceed with the stability analysis, we must have some information on the material constants arising in the stress tensor expression attached to each particular fluid considered. The results of theoretical and experimental research on fluids of grade n, lead to the fact that there is a connection between thermodynamics and stability of a considered flow, concluding that thermodynamic incompatibility implies instability. Therefore, we review here some arguments on the thermodynamic consistency of fluids of grade n, for $n = 1, 2, 3$. By consistency with thermodynamics, we understand that the material satisfies a dissipation principle and meets the requirement that the specific Helmholtz free energy be at its minimum value in the equilibrium.

Case n=1. In the case $n = 1$, given that

$$\mathbf{T} = -p\,\mathbf{I} + \mu\,\mathbf{A}_1, \tag{2.8}$$

with $\mu \geq 0$, we deal with the classical Navier-Stokes formula, which is regarded not as an approximate, but as an exact definition of a particular fluid, flows of which are not restricted to any approximating sequence. The stability for the Bénard problem in a fluid of first order with a constant viscosity has already been discussed in the first section of this thesis. It is briefly shown in one of the next sections that for a variable viscosity the stability analysis is dramatically changed. The extra terms arising in the energy equation lead to a more complicated energy to be considered in order to complete the nonlinear stability study.

Case n=2. When the constitutive assumption for the fluid of grade $n = 2$ is regarded as an exact model, namely when the stress tensor has the form

$$\mathbf{T} = -p\,\mathbf{I} + \mu\,\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.9}$$

then some constraints on the three material moduli, μ, α_1 and α_2 arise. Although the matter is not studied here, there was much confusion around this special class of fluids of differential type, confusion arising from the consequences of the thermodynamics and stability of this model. A detailed discussion on the status of fluids of second grade is given in Dunn & Fosdick [13] and Dunn & Rajagopal [14]. In fact, the various arguments were on the certain signs for the material moduli α_1 and α_2. These moduli have been determined experimentally for several different fluids by observing the stress response under certain special slow steady flows and then correlating the observations with the associated response function. It has first been shown by Green & Laws [29] that the thermodynamic restrictions imposed by the Clausius-Duhem inequality are:

$$\mu \geq 0, \quad \alpha_1 + \alpha_2 = 0.$$

More generally, the work of Dunn & Fosdick [13] motivated a sign for α_1, and proceeded to give a detailed analysis of the two complementary situations corresponding to $\alpha_1 \geq 0$ and $\alpha_1 < 0$. They have argued that in order for the second grade fluid equation to be compatible with thermodynamics and to have the Helmholtz free energy at its minimum value in the equilibrium, the material moduli must satisfy:

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{2.10}$$

Moreover, according to their analysis, in the case when α_1 is positive, the model has a good behaviour in the sense that stability and unboundedness may be achieved, whereas if α_1 is taken to be negative, then in quite arbitrary flows instability and boundedness are unavoidable. Therefore, they have concluded that the only second grade fluid to be found in nature is one with $\alpha_1 \geq 0$. Later, Fosdick & Rajagopal [19], have demonstrated that anomalous behaviour occurs if an exact model is assumed with

$$\mu \geq 0, \quad \alpha_1 < 0, \quad \alpha_1 + \alpha_2 \neq 0. \tag{2.11}$$

One anomalous behaviour is that for any closed sufficiently small but finite fixed rigid container which is filled with a fluid satisfying (2.9) under the restrictions (2.11), if the viscosity is large enough then for any initial disturbance at $t = 0$, the volume integral must become larger than any preassigned finite number at some time in $(0, \infty)$.

Their second result on the anomalous behaviour of any incompressible fluid that is assumed to be modelled by (2.9) and (2.11) is concerned with the temporal evolution of initial disturbances. They show that for any μ, α_1 and α_2 which satisfies (2.11), for any smooth initial velocity field disturbances for the same problem as above, and for all sufficiently small containers ω, the fluid motion caused by the initial disturbances can never subside.

We note here that throughout our study on the second grade fluid, we consider that the normal stress coefficients α_1 and α_2 satisfy the restrictions (2.10).

For certain applications to practical viscoelastic flows the above model of the second grade fluid has been recognised as inadequate. In particular, Man & Sun [43], apply viscoelastic fluid theory to the study of shear thickening and shear thinning. It seems that second order fluids fail to model these special effects. Therefore, a generalisation of the second grade fluid model has been adopted, of the form

$$\mathbf{T} = -p\,\mathbf{I} + \mu\,\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

with $\Pi = tr\mathbf{A}_1^2$ and m being a real number. The model is a combination of the classical power law viscoelastic fluid and that of a fluid of second grade.

Stability studies of the Bénard problem in fluids of second grade, respectively, generalised second grade, are presented in the following chapters. Due to the nonlinear terms present in the model, we expect to obtain nonlinear stability results of conditional type.

Case n=3. The fluid of third grade, namely when the stress relation is of the form

$$\mathbf{T} = -p\,\mathbf{I} + \mu\,\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\,\mathbf{A}_2)\,\mathbf{A}_1, \quad (2.12)$$

has been studied by Fosdick & Rajagopal [20] who have deduced that the constraints on the material moduli are:

$$\mu > 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\,\mu\beta_3}. \quad (2.13)$$

From the above relations, one can observe that when $\beta_3 = 0$, then $\alpha_1 + \alpha_2 = 0$ and the results of Dunn & Fosdick [13] hold.

When (2.13) holds, the base flow is stable relative to all disturbances that vanish at the boundary of the fluid domain, if the stretching and its diffusion are sufficiently small or if the viscosity is sufficiently large. Therefore, the result is a conditional criterion for nonlinear stability. The condition on both α_1 and α_2 plays an important role in the stability study, in the sense that, when the strict inequality holds then the energy decays exponentially, whereas for the case $|\alpha_1 + \alpha_2| = \sqrt{24\,\mu\beta_3}$ - which corresponds to the situation when the normal stress effects are comparable to the viscous effects - one is able to establish decay for the initial disturbances, but not necessarily exponentially.

As for the case when $\alpha_1 < 0$, the model is still compatible with the thermodynamics, but the free energy is not a minimum in equilibrium; it has been proved that for this situation there exist a very general class of mechanical isolated flows necessarily unbounded. The larger the viscosity is, holding everything fixed, the more rapidly the fluid motion becomes unbounded in time. These kinds of behaviour are similar with anomalous growth in the second order fluid study carried out by Fosdick & Rajagopal [19]. Therefore, negative α_1 gives rise to a fundamental asymptotic unstable theory of fluid behaviour.

We assume ahead that (2.13) holds. Due to the incompressibility condition (2.2), the trace of $\mathbf{A}_1$ is zero. Together, (2.13) and $tr\,\mathbf{A}_1 = 0$, are easily seen to imply that

$$\mathbf{T} = -p\,\mathbf{I} + \mu\,\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta(tr\,\mathbf{A}_1^2)\,\mathbf{A}_1, \tag{2.14}$$

where β denotes the material modulus β_3.

We investigate in one of the following chapters, the thermal convection in a layer of a third grade fluid with a constitutive relation (2.14), with viscosity depending on temperature variations. Due to the extra nonlinearity arising in (2.14), the nonlinear stability results are stronger than for the second grade fluid analysis. Moreover, the presence of the extra term related to the coefficient β enables us to provide unconditional nonlinear stability criteria using a natural energy approach.

2.2 Viscosity-temperature relation

We shall focus here on the stability of convection in some classes of generalised fluids whose viscosity varies with temperature. Much attention has recently been devoted to the subject of convection studies with variable viscosity, since viscosity is one of the fluid properties which may dramatically change with temperature; see e.g. Capone & Gentile [6]- [7], Depassier& Spiegel [12], Franchi & Straughan [23]- [25], Gertsberg & Sivashinsky [28], Oliver & Booker [48], Palm et al. [49], Richardson [53]- [54], Selak & Lebon [58], Straughan [61] and [65], Tippelskirch [69], Torrance & Turcotte [70] and see also several references given in Lide [42], page 6-158, regarding the dynamic viscosity of glycerin; while glycerin exhibits a dramatic viscosity change there are many other examples, oily fluids, quoted in the tables in Lide [42]. For olive oil, for example, the viscosity drops from 138 centipoise at 10°C to 12.4 centipoise at 70°C.

Even though most of the stability results for incompressible fluids are delivered on the assumption of a constant viscosity, Straughan [61] studied a viscosity model of Torrance & Turcotte [70] for the earth's mantle, and a porous medium model, allowing for variable viscosity, ν, of the form

$$\nu = \nu_0 \, e^{-\gamma z},$$

where ν_0, γ are constants and z is the vertical coordinate pointing upward through the fluid layer. Later, Richardson & Straughan [54] developed a nonlinear energy stability analysis for the case when ν is given by the linear relation of Palm et al. [49], namely,

$$\nu(T) = \nu_0 \, (1 - \gamma(T - T_0)), \tag{2.15}$$

with T_0, γ constants, or in terms of dynamic viscosity $\mu(T) = \nu(T)\rho_0$, where ρ_0 is a constant density for which

$$\mu(T) = \mu_0 \, (1 - \gamma(T - T_0))$$

with $\mu_0 = \nu_0\rho_0$. The same linear temperature dependent relation is used by Richardson & Straughan [55] in a porous medium, where they find need also for a Brinkman equation of momentum rather than a Darcy one. Richardson [53] has extended the linear relation to a quadratic one, for fluids like liquid sulphur which possess a viscosity maximum. The temperature dependency was considered to be

$$\nu(T) = \nu_0 \, (1 - \gamma(T - T_0)^2).$$

Franchi & Straughan [24]- [25] have used the Palm et. al. linear relation in stability studies for such fluids as generalised second grade, third grade and dipolar type. It is shown that some of these fluids (the third grade one and the dipolar) possess just the right kinds of dissipative terms to control the nonlinearities which arise when the viscosity varies with temperature and one can proceed with a natural energy, rather that generalised one. The nonlinear stability results achieved in their studies are conditional for all cases, but in the light of one of the next chapters we remark that an unconditional nonlinear stability result may be delivered for the third grade fluid when the viscosity is allowed to depend linearly on temperature.

The general problem of how the viscosity depends on the temperature has been considered by Capone & Gentile, and it has been pointed out that for most fluids the exponential dependence seems to be the most appropriate, according to the experimental results. The thermal convection for fluids with viscosity depending exponentially on temperature is studied by Capone & Gentile [6]; the exponential dependence is of the form

$$\nu(T) = \nu_0 \, e^{-\gamma(T-T_0)},$$

which is a realistic fit for many real fluids, as stated in Torrance & Turcotte [70]. It is shown that the condition assuring linear instability assures conditional nonlinear stability, too.

Tippelskirch [69] suggests another possible viscosity relation for a fluid, namely

$$\nu(T) = \frac{C}{1 + AT + BT^2}, \tag{2.16}$$

for A, B, C positive constants.

A nonlinear stability analysis for convection is presented by Straughan [66], when the viscosity may have a general dependence on temperature. A generalised energy analysis was necessary in order to deliver a conditional nonlinear stability result. Numerical calculations are given for a viscosity of the Tippelskirch form.

Much stronger nonlinear stability results are obtained by Payne & Straughan [51] for a flow in a porous media where the Forchheimer equations are considered. The problem of thermal convection in such a medium is studied when the viscosity has a linear form as in (2.15), and unconditional nonlinear stability is achieved. It is important to stress that the result was obtainable in L^3 or L^4 norms, instead of working with the L^2 norm.

In the next chapters, we focus on stability studies of convection in a layer of different generalized fluids, when the viscosity ν is allowed to have a general dependence on temperature, of the form

$$\nu(T) = \nu_0 f(T), \tag{2.17}$$

with $f(T)$ being a function of temperature T, or in terms of dynamic viscosity,

$$\mu(T) = \mu_0 f(T), \tag{2.18}$$

with $\mu_0 = \rho_0 \nu_0$.

Additionally, we assume that $f'(T)$ is bounded above by a positive constant

$$|f'(T)| \leq M. \tag{2.19}$$

The requirement is consistent with the physical properties for the viscosity of a fluid, as we may prove in the end of this section using some viscosity formulas of a selection of natural substances.

We consider the Taylor expansion

$$f(\bar{T} + \theta) = f(\bar{T}) + f'(\hat{T})\,\theta,$$

where $\bar{T}(z)$ is the temperature linear in z in the steady state, θ is the temperature perturbation and $\hat{T}$ arises through the remainder term in the Taylor expansion. If we denote $f(\bar{T}(z))$ by $F(z)$, then

$$f(\bar{T} + \theta) = F(z) + f'(\hat{T})\,\theta. \tag{2.20}$$

The generalisation (2.17) with the restriction (2.19) covers some viscosity functions found in the literature, such as Tippelskirch [69]. In addition to the results for the general viscosity, in some studies we provide numerical calculations for the viscosity in form (2.16) or for formulas used in Straughan [66], for the viscosity of aniline

$$\nu_1(T) = \frac{0.31482}{1 + 0.48727 \times 10^{-1} T + 0.87490 \times 10^{-3} T^2},$$

or for nitrobenzene

$$\nu_2(T) = \frac{2.6202}{1 + 0.26641 \times 10^{-1} T + 0.14832 \times 10^{-4} T^2}.$$

Note. We include here a brief note on some useful properties of the viscosity-temperature dependence, properties which play a relevant role in the following analyses.

We plot two functions of T, the formulas for the viscosity of aniline, and of nitrobenzene. As we need estimates for the function $F(z)$, we take the values of ν_1 and ν_2 at $\bar{T}$ and we collect them in Figure 2.1 and Figure 2.2. It is easy to see that the graphs remain always in the positive section, which essentially may be written

"*there exists a positive constant N, small enough, such that $F(z) > N$, for each value of z in* $(0,1)$."

If we represent the graphs of the first derivatives of ν_1 and ν_2 with respect of $\bar{T}$, Figure 2.3 and Figure 2.4, we observe that

"*there exists a positive value M such that $|\nu'(T)| < M$.*"

We conclude that the positive constants $N, M > 0$ such that $F(z) > N$ and, respectively $|f'(T)| < M$, arise naturally and so the assumptions that we use throughout the present work are consistent with physical reality.

Moreover, similar assumptions are considered in Flavin & Rionero work on convection, [15]- [18], where not only the viscosity but also the thermal diffusivity, $\kappa(T)$, depends in an arbitrary manner on the temperature, being bounded below by a positive constant, i.e.

$$\nu(T) \geq 1, \quad \text{and} \quad \kappa(T) \geq 1.$$

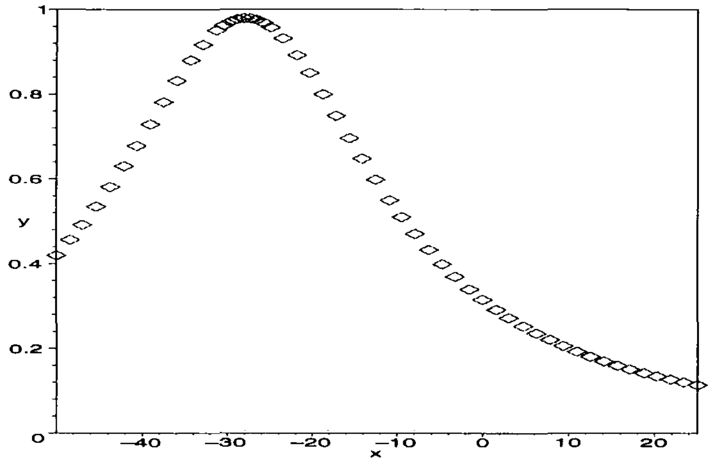

Figure 2.1: Viscosity of aniline, with $\bar{T}$ between $-50°C$ and $+25°C$

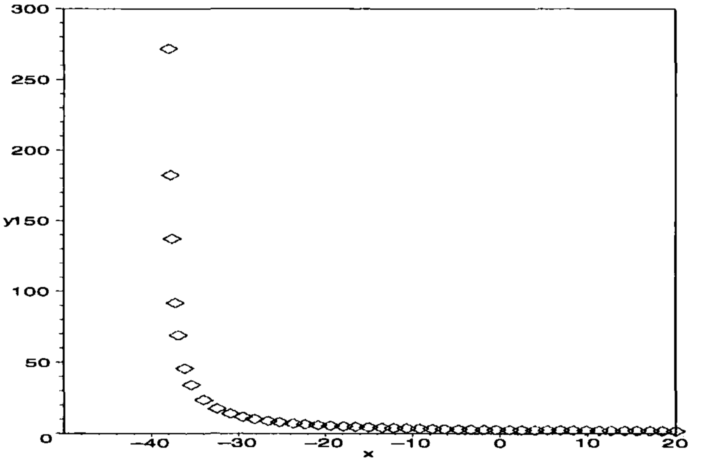

Figure 2.2: Viscosity of nitrobenzene, with $\bar{T}$ between $-50°C$ and $+25°C$

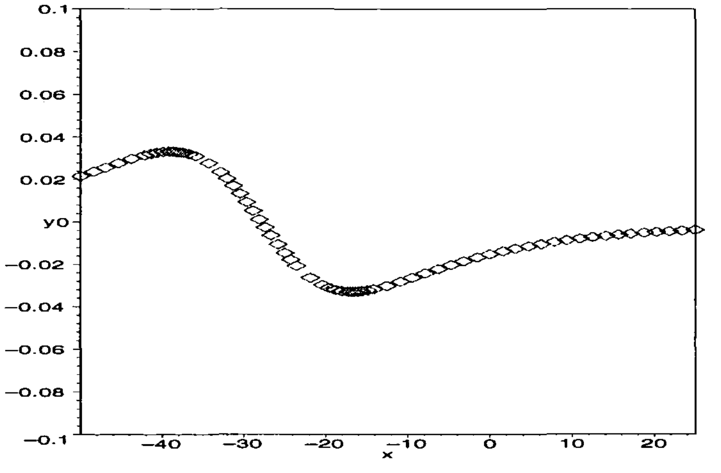

Figure 2.3: The first derivative of the viscosity of aniline, with $\bar{T}$ between $-50°C$ and $+25°C$

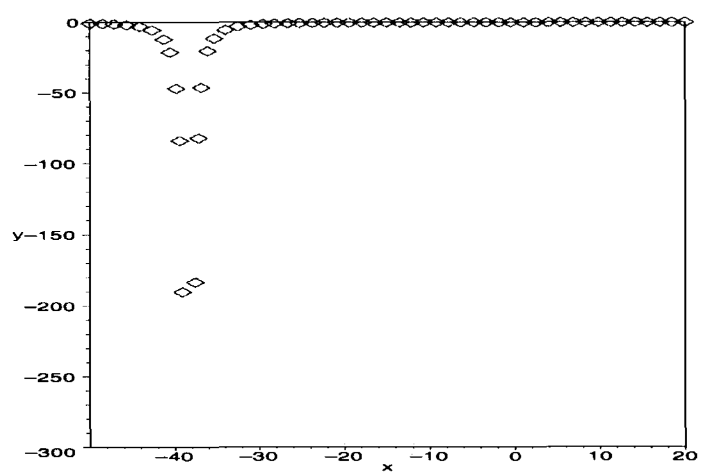

Figure 2.4: The first derivative of the viscosity of nitrobenzene, with $\bar{T}$ between $-50°C$ and $+25°C$

2.3 On stability for a Navier-Stokes fluid with $\nu(T)$

Amongst the many assumptions that we have made when studying the Bénard problem in a layer of a Navier-Stokes fluid, one was that the viscosity of the fluid has a constant value ν_0. In that simplified situation a natural energy was employed to derive an unconditional nonlinear stability criterion. Nevertheless, as we have already stated in the previous section, there is a temperature-viscosity relation which plays an important role in the stability behaviour of any particular fluid.

We do not intend here to present a fully nonlinear stability theory for a layer of a Navier-Stokes (first order) fluid with the viscosity being a general function of temperature, but for a complete picture of the stability for the viscoelastic fluids, we wish to highlight some important aspects of the analysis. We shall see why in the Navier-Stokes viscous stability analysis one cannot proceed with a natural energy method and what kind of analysis is necessary to be carried out in order to provide nonlinear stability results.

Let the viscosity-temperature relation be of form (2.17). The stress tensor of a fluid of first grade, (2.8), is then

$$\mathbf{T} = -p\,\mathbf{I} + \rho_0\nu(T)\,\mathbf{A}_1,$$

where

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T,$$

with $\mathbf{L}$ being the spatial gradient of velocity. The equations of motion (2.1)-(2.3) for an infinite horizontal Navier-Stokes fluid layer with $z \in (0, H)$ are rewritten as

$$\begin{aligned} \dot{v}_i &= -\frac{1}{\rho}p_{,i} + \{\nu(T)(A_1)_{ij}\}_{,j} - k_i g[1 - \alpha(T - T_0)], \\ v_{i,i} &= 0, \\ \dot{T} &= \kappa\Delta T, \end{aligned} \tag{2.21}$$

where ρ, ν, g, α, T, κ are respectively, density, kinematic viscosity, gravity, thermal expansion coefficient, temperature, thermal diffusivity and $\mathbf{k}$ is the vector $(0, 0, 1)$.

We treat here $\rho\,(=\rho_0)$ as constant. On the boundaries

$$\begin{aligned} v_i &= 0, \quad z=0,H, \\ T &= T_0, \quad z=0, \\ T &= T_H, \quad z=H,, \end{aligned} \tag{2.22}$$

with $T_0 > T_H$.

Then the steady solution whose stability we investigate is $(\bar{v}_i, \bar{T}, \bar{p})$, with $\bar{v}_i \equiv 0$, $\bar{T} = -\zeta\, z + T_0$ and ζ being the temperature gradient given by

$$\zeta = \frac{\Delta T}{H}, \quad \Delta T = T_0 - T_H > 0.$$

Perturbations (v_i, T, p) of $(\bar{v}_i, \bar{T}, \bar{p})$ are introduced via

$$v_i = \bar{v}_i + u_i, \quad T = \bar{T} + \theta, \quad p = \bar{p} + \pi,$$

and the dimensionless perturbed equations are

$$\begin{aligned} \dot{u}_i &= -\pi_{,i} + \delta_{i3} R\theta + \{f(\bar{T}+\theta)a_{ij}\}_{,j}, \\ u_{i,i} &= 0, \\ Pr\,\dot{\theta} &= \Delta\theta + Rw, \end{aligned}$$

with $z \in (0,1)$, or equivalently,

$$\begin{aligned} \dot{u}_i &= -\pi_{,i} + \delta_{i3} R\theta + \{F(z)a_{ij}\}_{,j} + \{f'(\hat{T})\theta\, a_{ij}\}_{,j}\,, \\ u_{i,i} &= 0, \\ Pr\,\dot{\theta} &= \Delta\theta + Rw, \end{aligned} \tag{2.23}$$

where (2.20) has been employed and scalings similar to those from the introductory part were considered. Here $w = u_3$, $a_{ij} = u_{i,j} + u_{j,i}$, and the spatial domain has become

$$\{(x,y) \in \mathbb{R}^2\} \times \{z \in (0,1)\}.$$

Though we do not develop a complete nonlinear stability analysis for the steady solution of (2.21), we briefly show how the general dependence of the viscosity on the temperature may change the stability analysis.

We form the energy identities by multiplying $(2.23)_1$ by u_i, $(2.23)_3$ by θ and then integrate over a periodic cell V to find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\mathbf{u}||^2 &= R\langle \theta\, w\rangle - \frac{1}{2}\langle F(z)\, a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta\, a_{ij}a_{ij}\rangle, \\ \frac{Pr}{2}\frac{d}{dt}||\theta||^2 &= R\langle \theta\, w\rangle - ||\nabla\theta||^2. \end{aligned} \tag{2.24}$$

Then, considering the combination $(2.24)_1 + \lambda(2.24)_2$, with λ being a positive constant to be chosen later, the energy identity obtained is of form

$$\frac{dE}{dt} = R\mathcal{I} - \mathcal{D} - \frac{1}{2}\langle f'(\hat{T})\theta\, a_{ij}a_{ij}\rangle, \tag{2.25}$$

where

$$E(t) = \frac{1}{2}\left[\, ||\mathbf{u}||^2 + \lambda Pr||\theta||^2\right]$$

is the natural energy arising from (2.24) and

$$\begin{aligned} \mathcal{I} &= (1+\lambda)\,\langle \theta\, w\rangle, \\ \mathcal{D} &= \lambda||\nabla\theta||^2 + \frac{1}{2}\langle F(z)\, a_{ij}a_{ij}\rangle. \end{aligned}$$

The next step is to define the critical energy stability number R_E by

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}},$$

where $\mathcal{H}$ is the space of admissible solutions we are working with.

The nonlinear energy stability threshold requires

$$R < R_E,$$

thus $a = (R_E - R)/R_E > 0$. From (2.25) we may derive

$$\frac{dE}{dt} \le -a\,\mathcal{D} - \frac{1}{2}\langle f'(\hat{T})\theta\, a_{ij}a_{ij}\rangle. \tag{2.26}$$

An estimation of the cubic term

$$\mathcal{N} = -\frac{1}{2}\langle f'(\hat{T})\theta\, a_{ij}a_{ij}\rangle$$

is required in order to proceed with a nonlinear stability analysis. We first employ the estimate (2.19) to have

$$\mathcal{N} = -\frac{1}{2}\langle f'(\hat{T})\theta\, a_{ij}a_{ij}\rangle \le \frac{M}{2}\langle |\theta|\, a_{ij}a_{ij}\rangle. \tag{2.27}$$

To this stage, Cauchy's inequality proves to be inefficient as the energy form is not strong enough to handle the resulting terms.

One idea is to make use of the result, that there exists a positive constant c such that

$$\sup_V |\theta| \leq c\,||\Delta\theta||. \tag{2.28}$$

The inequality for the supremum of a function θ with $\theta|_{z=0} = \theta|_{z=1} = 0$ (it is proven to work for $\mathbf{u}$, as well) is very important in the present analysis and furthermore, throughout the following work. The positive constant c depends on the geometry of the domain V. A complete proof for (2.28) is available in Galdi & Straughan [27] or Straughan [64], Appendix A.

Upon using (2.28) in (2.27), the bound for $\mathcal{N}$ is found to be

$$\mathcal{N} \leq \frac{M}{2} \sup_V |\theta|\, ||a_{ij}||^2 \leq M\, c\,||\Delta\theta||\, ||\nabla\mathbf{u}||^2,$$

and from (2.26) we find

$$\frac{dE}{dt} \leq -a\mathcal{D} + M\, c\,||\Delta\theta||\, ||\nabla\mathbf{u}||^2. \tag{2.29}$$

The idea now is to estimate the cubic term from (2.29) in terms of $\mathcal{D}E^{1/2}$. It is easy to see that neither $||\Delta\theta||$ nor $||\nabla\mathbf{u}||^2$ are present in the definitions of the natural energy, nor is $||\Delta\theta||$ in the dissipative term $\mathcal{D}$. This suggests that another energy might include both terms $||\Delta\theta||^2$ and $||\nabla\mathbf{u}||^2$. A *generalised energy* is then defined by

$$E(t) = \frac{1}{2}\,[\,||\mathbf{u}||^2 + \lambda Pr||\theta||^2 + \gamma||\Delta\theta||^2 + \alpha||\nabla\mathbf{u}||^2],$$

with γ, α being positive coupling parameters.

Richardson [53], Chapters 3-6, presents a full discussion on stability analyses (linear and nonlinear) for a Navier-Stokes fluid heated from below, allowing the viscosity to have a linear or a quadratic dependence on the temperature. Though the study provides that the nonlinear stability boundary coincides with the linear instability one, the nonlinear stability criterion is a conditional one due to the nonlinearities in $\mathcal{N}$. An analysis with a general viscosity-temperature relation was carried out in Straughan [66], with similar results.

Chapter 3

Nonlinear stability for the second grade fluid

The thermal convection in a layer of fluid of second grade is investigated, with the viscosity being a general function of temperature. We address the nonlinear stability analysis and prove that *conditional nonlinear stability* may be achieved. A generalised energy analysis is found to be necessary to investigate the nonlinear stability for the Bénard problem.

This chapter is submitted for publication (Budu [4]).

3.1 The convection equations for a second grade fluid

The model for a layer of fluid of second grade between two horizontal plates, heated from below, consists of the momentum equation,

$$\rho \dot{v}_i = \rho f_i + T_{ji,j}, \tag{3.1}$$

the continuity equation,

$$v_{i,i} = 0, \tag{3.2}$$

and the balance of energy equation,

$$\rho \dot{\varepsilon} = T_{ij} L_{ij} - q_{i,i} + \rho r, \tag{3.3}$$

where v_i, ρ, f_i, $\mathbf{T}$, r, ε, q_i and $\mathbf{L}$ are respectively, the velocity, density, body force, stress tensor, heat supply, internal energy, heat flux and velocity gradient. Standard indicial notation is used and a superposed dot denotes the material derivative.

The stress tensor associated to the second grade fluid is of form (2.9), namely

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2$$

where $\mathbf{A}_1 = [A_{ij}]$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors, defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1,$$

$\mathbf{L}$ being the velocity gradient. We shall assume the normal stress coefficients α_1, α_2 are constants satisfying the restrictions (2.10), cf. Dunn & Fosdick [13]

$$\alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0.$$

Throughout we shall refer to a general temperature dependent viscosity, as stated in (2.17).

For the Bénard convection problem it is sufficient to set here $r = 0$, then we may follow the analysis of Franchi & Straughan [22] and reduce the balance of energy equation (3.3) to

$$\dot{T} = \kappa\Delta T,$$

with Δ being the Laplacian operator and κ the thermal diffusivity. We employ a Boussinesq approximation, so that $\rho = \rho_0$ (constant) everywhere except in the body force term of (3.1), for which

$$\rho f_i = -\rho_0 g\delta_{i3}\left[1 - \alpha(T - T_0)\right].$$

Here g is gravity, T_0 is a reference temperature, and α is the coefficient of thermal expansion.

Suppose the fluid is contained in the infinite horizontal layer with $z \in (0, H)$, then we may show that the governing equations (3.1)-(3.3) reduce to

$$
\begin{aligned}
\rho_0 \dot{v}_i &= -p_{,i} - \rho_0 g \delta_{i3} \left[1 - \alpha(T - T_0)\right] + \{\mu(T) A_{ij}\}_{,j} \\
&\quad + \alpha_1 (\dot{A}_{ij} + v_k A_{ij,k} + A_{im} L_{mj} + L_{mi} A_{mj})_{,j} \\
&\quad + \alpha_2 \left(A_{im} A_{mj}\right)_{,j} , \\
v_{i,i} &= 0, \\
\dot{T} &= \kappa \Delta T.
\end{aligned}
\tag{3.4}
$$

If the top and the bottom surfaces are regarded as no-slip boundaries, then the velocity vector field boundary conditions are

$$
v_i = 0, \quad z = 0, H. \tag{3.5}
$$

Further, the temperature boundary conditions are

$$
T = T_0, \quad z = 0 \quad \text{and} \quad T = T_H, \quad z = H \tag{3.6}
$$

with T_0, T_H being fixed temperature values and $T_0 > T_H$ (hotter below).

A steady solution for the boundary problem (3.4)-(3.6) is $(\bar{v}_i, \bar{T}, \bar{p})$, with $\bar{v}_i \equiv 0$ and $\bar{T}$ a function of z. From (3.4) it then follows

$$
\begin{aligned}
0 &= -\bar{p}_{,i} - \rho_0 g \delta_{i3} \left[1 - \alpha(\bar{T} - T_0)\right] \\
\bar{v}_{i,i} &= 0, \\
0 &= \kappa \bar{T}_{zz}.
\end{aligned}
\tag{3.7}
$$

From $(3.7)_3$, using the boundary conditions (3.6),

$$
\bar{T} = -\zeta z + T_0, \tag{3.8}
$$

where $\zeta = (T_0 - T_H) H^{-1}$. From $(3.7)_1$

$$
\bar{p} = -\rho_0 g\, z - \frac{\rho_0 g\, \alpha \, \zeta}{2}\, z^2 + p_0, \tag{3.9}
$$

is the steady pressure field with p_0 being a constant.

To study the nonlinear stability of the steady solution we let $(u_i,\, \theta,\, \pi)$ be perturbations to $(\bar{v}_i,\, \bar{T},\, \bar{p})$. The resulting perturbation equations are non-dimensionalized via

$$
\begin{gathered}
x_i = x_i^* H, \quad u_i = u_i^* U, \quad \pi = \pi^* P, \quad \theta = \theta^* \tilde{T}, \quad t = t^* \mathcal{T}, \\
U = \frac{\nu_0}{H}, \quad P = \frac{U \rho_0 \nu_o}{H}, \quad \tilde{T} = U\sqrt{\frac{Pr\,\zeta}{\alpha g}}, \quad \mathcal{T} = \frac{H^2}{\nu_0}, \\
Pr = \frac{\nu_0}{\kappa}, \quad R = \sqrt{\frac{\alpha \zeta g H^4}{\nu_0 \kappa}}, \quad \Gamma_1 = \frac{\rho_0 H^2}{\alpha_1}, \quad \Gamma_2 = \frac{\rho_0 H^2}{\alpha_2},
\end{gathered}
$$

where $Ra = R^2$, Pr are the Rayleigh and Prandtl numbers and Γ_1, Γ_2 are absorption numbers.

Omitting all stars, the non-dimensional perturbed equations, for $(x, y) \in \mathbb{R}^2$ and $z \in (0,1)$, become:

$$
\begin{aligned}
\dot{u}_i &= -\pi_{,i} + \delta_{i3} R\theta + \{f(\bar{T}+\theta) a_{ij}\}_{,j} \\
&\quad + \frac{1}{\Gamma_1}\big(a_{ij,t} + u_k a_{ij,k} + a_{im} u_{m,j} + u_{m,i} a_{mj}\big)_{,j} \\
&\quad + \frac{1}{\Gamma_2}\big(a_{im} a_{mj}\big)_{,j}, \\
u_{i,i} &= 0, \\
Pr\dot{\theta} &= \Delta\theta + Rw,
\end{aligned} \tag{3.10}
$$

where $w = u_3$ and $a_{ij} = u_{i,j} + u_{j,i}$.

The corresponding boundary conditions are

$$
u_i = \theta = 0, \quad z = 0, 1, \tag{3.11}
$$

with u_i, θ and π having a periodic shape in (x, y).

Taking into account the Taylor expansion (2.20), equations (3.10) may be written

$$
\begin{aligned}
\dot{u}_i &= -\pi_{,i} + \delta_{i3} R\theta + \{F(z) a_{ij}\}_{,j} + \{f'(\hat{T})\theta a_{ij}\}_{,j} \\
&\quad + \frac{1}{\Gamma_1}\big(a_{ij,t} + u_k a_{ij,k} + a_{im} u_{m,j} + u_{m,i} a_{mj}\big)_{,j} \\
&\quad + \frac{1}{\Gamma_2}\big(a_{im} a_{mj}\big)_{,j}. \\
u_{i,i} &= 0, \\
Pr\dot{\theta} &= \Delta\theta + Rw.
\end{aligned} \tag{3.12}
$$

3.2 Conditional nonlinear stability analysis

We now form the main energy identities. Multiply $(3.12)_1$ by u_i, $(3.12)_3$ by θ and integrate over the period cell V. After use of integration by parts and the boundary conditions, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2] &= R\langle\theta\, w\rangle - \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle \quad (3.13)\\ \frac{1}{2}\frac{d}{dt}Pr||\theta||^2 &= R\langle\theta\, w\rangle - ||\nabla\theta||^2. \quad (3.14)\end{aligned}$$

We now form the combination (3.13)+λ(3.14), with λ being a positive parameter

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2 + \lambda Pr||\theta||^2] &= R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2\\ &- \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle,\end{aligned}$$

and define the natural energy by

$$E(t) = \frac{1}{2}[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2 + \lambda Pr||\theta||^2].$$

Taking into account the bound for the first derivative of the viscosity, (2.19), it follows that

$$\begin{aligned}\frac{dE}{dt} &\leq R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2\\ &- \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle + \frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle. \quad (3.15)\end{aligned}$$

We next introduce

$$\frac{1}{R_E} = \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}, \quad (3.16)$$

where $\mathcal{H}$ is the space of admissible functions over which the maximum is sought and

$$\begin{aligned}\mathcal{I} &= (1+\lambda)\langle\theta\, w\rangle,\\ \mathcal{D} &= \lambda||\nabla\theta||^2 + \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle.\end{aligned}$$

Here $\mathcal{H}$ is the space of functions

$$\begin{aligned}\mathcal{H} = \{&u_i,\theta\,|\,u_i\in W^{1,2}(V), u_{i,i}=0, u_i=0 \text{ at } z=0,1;\\ &\theta\in W^{1,2}(V), \theta=0 \text{ at } z=0,1\}\end{aligned}$$

u_i, θ satisfying a plane tiling periodic planform in x and y.

We do not calculate R_E here, but we observe that the linearised form of (3.12) is symmetric and so the critical Rayleigh number R_E is just that of linear instability theory, R_L. A standard numerical calculation will yield R_L.

However, here the attention is focused on developing a nonlinear analysis. From (3.15) we derive

$$\frac{dE}{dt} \leq R\mathcal{I} - \mathcal{D} + \frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle = -\mathcal{D}\,(1 - R\frac{\mathcal{I}}{\mathcal{D}}) + \frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle.$$

Recalling further the definition (3.16) we obtain

$$\frac{dE}{dt} \leq -\mathcal{D}\,(1 - \frac{R}{R_E}) + \frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle.$$

We require that $R < R_E$, to conclude that

$$\frac{dE}{dt} \leq -a\mathcal{D} + \frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle, \tag{3.17}$$

where $a = (R_E - R)/R_E > 0$.

The remaining problem is to handle the nonlinearity in (3.17). To this end, we remark that use of the Cauchy inequality to write

$$\frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle \leq \frac{M}{2}||\theta||\,||a_{ij}a_{ij}||,$$

leads to difficulties with the $||a_{ij}a_{ij}||$ term. Though we may handle the $||\theta||$ term from the expression of E, by Poincaré's inequality, the nonlinear term arising above cannot be controlled only with the $\langle a_{ij}a_{ij}\rangle$ term from $\mathcal{D}$.

To overcome this, we follow a similar procedure as for a Navier-Stokes fluid analysis. We employ the embedding inequality, (2.28),

$$\sup_V |\theta| \leq c\,||\Delta\theta||,$$

to deduce that

$$\begin{aligned}\frac{M}{2}\langle|\theta|a_{ij}a_{ij}\rangle &\leq \frac{M}{2}\sup_V|\theta|\,||a_{ij}||^2 \\ &\leq M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2. \end{aligned}\tag{3.18}$$

We remark that the natural energy would not be enough to control the nonlinearities in (3.18), as the $||\Delta\theta||$ term does not appear in the expression of E. A *generalised* *energy* analysis is found to be necessary to investigate fully nonlinear stability. The natural presence of a $||\nabla \mathbf{u}||^2$ term in E is essential in what follows. In the Navier-Stokes fluid analysis such a term had to be added artificially, see the last section, and the analysis is then more complicated.

The inequality (3.18) suggests that a suitable generalised energy might include a $||\Delta\theta||^2$ term. With this aim, we take the Laplacian of $(3.10)_3$ to find

$$Pr\,\Delta\theta_{,t} + Pr\,(\Delta u_i\theta_{,i} + 2u_{i,j}\theta_{,ij} + u_i\Delta\theta_{,i}) \;=\; R\Delta w + \Delta^2\theta, \tag{3.19}$$

and we also conclude from (3.10) that

$$\Delta\theta = 0, \quad \text{on} \quad z = 0, 1.$$

We multiply (3.19) by $\Delta\theta$ and integrate over V to obtain

$$\begin{aligned}\frac{1}{2}Pr\frac{d}{dt}||\Delta\theta||^2 \;=\; & -\; Pr\,(\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle + \langle\Delta\theta\,u_i\,\Delta\theta_{,i}\rangle)\\ & -\; ||\nabla\,\Delta\theta||^2 \;+\; R\langle\Delta w\,\Delta\theta\rangle.\end{aligned} \tag{3.20}$$

We define now the new *generalised energy* as

$$\mathcal{E}(t) \;=\; E(t) + \frac{\gamma}{2}Pr||\Delta\theta||^2,$$

with $\gamma(>0)$ a parameter to be chosen later. The new energy inequality is obtained by adding $\gamma \times (3.20)$ to (3.17) with the result

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \;\leq\; & -\; a\,\mathcal{D} - \gamma||\nabla\,\Delta\theta||^2 \;+\; \gamma R\langle\Delta w\,\Delta\theta\rangle + M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2\\ & -\; \gamma Pr\,[\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle + \langle\Delta\theta\,u_i\,\Delta\theta_{,i}\rangle],\end{aligned} \tag{3.21}$$

where the estimation (3.18) has also been considered.

One can see that new nonlinear cubic terms have arisen. To control these terms we employ the analysis below.

First, the term $\langle\Delta\theta\,u_i\,\Delta\theta_{,i}\rangle$ is zero under the assumptions of the boundary conditions considered here . We further observe that integration by parts leads to

$$\begin{aligned}[\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle] \;&=\; [-\langle\Delta\theta_{,j}\,u_{i,j}\,\theta_{,i}\rangle + \langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle]\\ &=\; [-\langle\Delta\theta_{,j}\,u_{i,j}\,\theta_{,i}\rangle - \langle\Delta\theta_{,i}\,u_{i,j}\,\theta_{,j}\rangle].\end{aligned} \tag{3.22}$$

For an estimation of the cubic terms in the square brackets, we do need a result for $\sup_V |\nabla\theta|$, cf. Franchi & Straughan [25] or Richardson [53], which states that there exists a positive constant c_1 such that

$$\sup_V |\nabla\theta| \leq c_1 \,||\nabla\,\Delta\theta||. \tag{3.23}$$

A complete proof of (3.23) and suitable values for the positive constant c_1 can be found in Adams [1] (p.99 and on).

Upon using (3.23) and the Poincaré inequality on the cubic terms of (3.22), it follows that

$$-\langle\Delta\theta_{,j}\, u_{i,j}\,\theta_{,i}\rangle - \langle\Delta\theta_{,i}\, u_{i,j}\,\theta_{,j}\rangle \leq\; 2 \sup_V |\nabla\,\theta|\, ||\nabla\,\Delta\theta||\, ||\nabla\mathbf{u}|| \leq\; 2\,c_1\, ||\nabla\mathbf{u}||\, ||\nabla\,\Delta\theta||^2.$$

We may conclude that

$$-\gamma Pr\,[\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\, u_{i,j}\,\theta_{,ij}\rangle] \;\leq\; 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2, \tag{3.24}$$

with $c_1 > 0$ suitably interpreted.

It then follows from (3.21) that

$$\begin{aligned} \frac{d\mathcal{E}}{dt} \leq \; & - \;\; a\,\mathcal{D} - \gamma||\nabla\,\Delta\theta||^2 + \gamma R\langle\Delta w\,\Delta\theta\rangle \\ & + \;\; M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2 + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2. \end{aligned} \tag{3.25}$$

We handle the term $\gamma R\langle\Delta w\,\Delta\theta\rangle$ with the aid of integration by parts and the arithmetic-geometric mean inequality to find

$$\gamma R\langle\Delta w\,\Delta\theta\rangle = -\gamma R\langle\nabla w\,\nabla\,\Delta\theta\rangle \leq \frac{\gamma R}{2\alpha}||\nabla w||^2 + \frac{\gamma R\alpha}{2}||\nabla\,\Delta\theta||^2, \tag{3.26}$$

where α is a suitable positive constant. Employing (3.26) in (3.25), one may deduce from the energy inequality

$$\begin{aligned} \frac{d\mathcal{E}}{dt} \leq \; & - \;\; a\mathcal{D} + \frac{\gamma R}{2\alpha}||\nabla w||^2 + \frac{\gamma R\alpha}{2}||\nabla\,\Delta\theta||^2 - \gamma||\nabla\,\Delta\theta||^2 \\ & + \;\; M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2 + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2. \end{aligned} \tag{3.27}$$

We seek an exponential decay for the energy, therefore we try to use the negative terms from (3.27), $-a\mathcal{D}$ and $-\gamma||\nabla\,\Delta\theta||^2$, to dominate the positive quantities arising from (3.26).

We first choose $\alpha = 1/R$, such that

$$\frac{\gamma R\alpha}{2} - \gamma = -\frac{\gamma}{2}.$$

Hence

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \le &- a\mathcal{D} + \frac{\gamma R^2}{2}||\nabla w||^2 - \frac{\gamma}{2}||\nabla\,\Delta\theta||^2\\ &+ M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2 + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2.\end{aligned} \tag{3.28}$$

Next we observe that

$$\frac{\mathcal{D}}{2} = \frac{\lambda}{2}||\nabla\theta||^2 + \frac{1}{4}\langle F(z)\,a_{ij}a_{ij}\rangle,$$

therefore

$$\frac{\mathcal{D}}{2} \ge \frac{1}{4}\langle F(z)\,a_{ij}a_{ij}\rangle.$$

As $F(z) > N$, for a small enough $N > 0$, then

$$\frac{\mathcal{D}}{2} \ge \frac{1}{4}\langle F(z)\,a_{ij}a_{ij}\rangle \ge \frac{N}{4}||a_{ij}||^2 = \frac{N}{2}||\nabla\mathbf{u}||^2,$$

and eventually

$$-a\mathcal{D} \le -\frac{a}{2}\mathcal{D} - \frac{a\,N}{2}||\nabla\mathbf{u}||^2. \tag{3.29}$$

Upon using (3.29) in (3.28) it turns out that

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \le &- \frac{a}{2}\mathcal{D} - (\frac{a\,N}{2} - \frac{\gamma R^2}{2})||\nabla\mathbf{u}||^2 - \frac{\gamma}{2}||\nabla\,\Delta\theta||^2\\ &+ M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2 + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2.\end{aligned}$$

We take $\gamma = a\,N/R^2$ to have

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \le &- \frac{a}{2}[\mathcal{D} + \frac{N}{R^2}||\nabla\,\Delta\theta||^2] + M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2\\ &+ 2\frac{aNc_1Pr}{R^2}\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2,\end{aligned} \tag{3.30}$$

or, equivalently

$$\frac{d\mathcal{E}}{dt} \le -\hat{\mathcal{D}} + \hat{\mathcal{N}}, \tag{3.31}$$

where

$$\hat{\mathcal{D}} = \frac{a}{2}\,\{\lambda||\nabla\theta||^2 + \frac{1}{2}\langle F(z)\,a_{ij}a_{ij}\rangle + \frac{N}{R^2}||\nabla\,\Delta\theta||^2\},$$

$$\hat{\mathcal{N}} = M\,c||\Delta\theta||\,||\nabla\mathbf{u}||^2 + \frac{2a\,Nc_1Pr}{R^2}\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2.$$

We may now provide a bound for the nonlinear term $\hat{\mathcal{N}}$ in terms of $\hat{\mathcal{D}}\mathcal{E}^q$, as the generalised energy $\mathcal{E}$ has the right properties to control the nonlinear terms of $\hat{\mathcal{N}}$. For each of the cubic terms it follows

$$\begin{aligned} M\, c||\Delta\theta||\,||\nabla \mathbf{u}||^2 &\leq \frac{2\sqrt{2}\, M\, c\, R}{a\, N \cdot \sqrt{a\, N\, Pr}} \mathcal{E}^{1/2}\hat{\mathcal{D}}, \\ \frac{2a\, N c_1 Pr}{R^2}||\nabla \mathbf{u}||\,||\nabla\, \Delta\theta||^2 &\leq 4\sqrt{2}c_1 Pr\sqrt{\Gamma_1}\mathcal{E}^{1/2}\hat{\mathcal{D}}, \end{aligned}$$

where Poincaré's inequality has also been used. The estimation for $\hat{\mathcal{N}}$ is then

$$\hat{\mathcal{N}} \leq A\mathcal{E}^{1/2}\hat{\mathcal{D}}, \tag{3.32}$$

with

$$A = \frac{2\sqrt{2}\, M\, c\, R}{a\, N \cdot \sqrt{a\, N\, Pr}} + 4\sqrt{2}c_1 Pr\sqrt{\Gamma_1} > 0.$$

We employ (3.32) in (3.31) to obtain

$$\frac{d\mathcal{E}}{dt} \leq -\hat{\mathcal{D}}(1 - A\mathcal{E}^{1/2}). \tag{3.33}$$

Upon using the Poincaré inequality we show that there exists a positive constant ψ such that $\hat{\mathcal{D}} \geq \psi\, \mathcal{E}$, hence

$$\frac{d\mathcal{E}}{dt} \leq -\psi\, \mathcal{E}\, (1 - A\mathcal{E}^{1/2}). \tag{3.34}$$

Employing a continuity argument on $\mathcal{E}(t)$, (3.33) ensures nonlinear stability as long as

$$(a)\, R < R_E, \qquad (b)\, \mathcal{E}^{1/2}(0) < \frac{1}{A}. \tag{3.35}$$

To prove the last statement, we define $k = 1 - A\mathcal{E}^{1/2}(0) > 0$, i.e. (3.35)(b) holds. There are now two possibilities, either

$$\mathcal{E}^{1/2}(t) < \frac{1}{A}, \quad \text{for any} \quad t \geq 0 \tag{3.36}$$

or there exists an $\eta > 0$ such that $\mathcal{E}^{1/2}(\eta) = \frac{1}{A}$, and $\mathcal{E}^{1/2}(t) < \frac{1}{A}$, for any $t \in [0, \eta)$. Therefore, from (3.34) it follows that

$$\frac{d\mathcal{E}}{dt} \leq 0, \quad \text{for} \quad t \in [0, \eta),$$

hence, $\mathcal{E}^{1/2}(t) \leq \mathcal{E}^{1/2}(0) < \frac{1}{A}$, for any $t \in [0, \eta)$. As $\mathcal{E}(t)$ is a continuous function of t on $[0, \eta]$, then it is impossible that $\mathcal{E}^{1/2}(\eta) = \frac{1}{A}$. This contradiction leads to (3.36), and consequently

$$\frac{d\mathcal{E}}{dt} \leq -\psi\,\mathcal{E}\,(1 - A\mathcal{E}^{1/2}(t)) < 0, \quad t \geq 0.$$

Hence

$$\mathcal{E}^{1/2}(t) \leq \mathcal{E}^{1/2}(0),$$

for any $t \geq 0$, and from (3.33) it follows that

$$\frac{d\mathcal{E}}{dt} \leq -\psi\,\mathcal{E}\,(1 - A\mathcal{E}^{1/2}(t)) \leq -\psi\,\mathcal{E}\,(1 - A\mathcal{E}^{1/2}(0)) = -k\,\psi\,\mathcal{E}(t).$$

Integrating the last inequality one may get

$$\mathcal{E}(t) \leq \mathcal{E}(0)\,e^{-k\psi t},$$

which essentially implies that $\mathcal{E}(t) \longrightarrow 0$, as $t \longrightarrow \infty$.

Concluding remark. What we have established is that provided $R < R_E$ and the initial energy satisfies the threshold (3.35)(b), then *conditional nonlinear stability* is achieved for a fluid of second grade heated from below, when the viscosity is a general function of temperature. We remark that a generalised energy was employed in order to determine a boundary for the stability, though the analysis is less complicated than the one for the first grade fluid due to the presence of the $||\nabla \mathbf{u}||^2$ term in the natural energy.

Chapter 4

Nonlinear stability for the generalised second grade fluid

The thermal convection in a layer of a generalised fluid of second grade is investigated, with the viscosity being a general function of temperature. We attack the nonlinear stability analysis and prove that *conditional nonlinear stability* is achieved. A generalised energy analysis is found to be necessary to investigate the nonlinear stability for the Bénard problem utilising the constitutive theory of Man & Sun [43].

This chapter is submitted for publication (Budu [4]).

4.1 The convection equations for a generalised second grade fluid

Man & Sun [43] adopt a generalisation of the stress tensor for the second grade fluid, to the study of glacier flows and they require a model capable of producing the effects of shear thickening and shear thinning. Hence, the generalised tensor considered is of form

$$\mathbf{T} = -p\mathbf{I} + \mu\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{4.1}$$

where $\Pi = tr\,\mathbf{A}_1^2$, m is a real number and $\mathbf{A}_1$, $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors given by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{A}_2 = \dot{\mathbf{A}_1} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1,$$

with $\mathbf{L}$ being the spatial gradient of velocity. In [43] they choose $\Pi = \frac{1}{2} tr\,\mathbf{A}_1^2$, but there is no mathematical loss in absorbing the $\frac{1}{2}$ in μ. This model is thus a combination of the classical power law viscoelastic fluid and of a fluid of second grade. If $\mu\Pi^{m/2}$ is regarded as the viscosity, then (4.1) clearly exhibits shear thickening when $m > 0$ - i.e., the viscosity increases with increasing velocity shear, whereas when $m < 0$, shear thinning is predicted, i.e., the viscosity decreases with increasing velocity shear.

The equations of motion are, the momentum equation,

$$\rho\dot{v}_i = \rho f_i + T_{ji,j}, \tag{4.2}$$

the continuity equation,

$$v_{i,i} = 0,$$

and the balance of energy,

$$\dot{T} = \kappa\Delta T,$$

where v_i, ρ, f_i, $\mathbf{T}$, r, ε and q_i are respectively, the velocity, density, body force, stress tensor, heat supply, internal energy and heat flux. Here Δ is the Laplacian operator and κ the thermal diffusivity. We employ a Boussinesq approximation so that $\rho = \rho_0$ (constant) everywhere except in the body force term in (4.2), for which

$$\rho f_i = -\rho_0 g\delta_{i3}\left[1 - \alpha(T - T_0)\right]$$

where g is gravity, T_0 is a reference temperature, and α is the coefficient of thermal expansion.

Considering once again that the fluid is contained in the infinite horizontal layer $z \in (0, H)$, the relevant equations of motion are

$$
\begin{aligned}
\rho_0 \dot{v}_i &= -p_{,i} - \rho_0 g \delta_{i3}\left[1 - \alpha(T - T_0)\right] + \{\mu(T)\Pi^{m/2}A_{ij}\}_{,j} \\
&\quad + \alpha_1(\dot{A}_{ij} + v_k A_{ij,k} + A_{im}L_{mj} + L_{mi}A_{mj})_{,j} \\
&\quad + \alpha_2\,(A_{im}A_{mj})_{,j}\,, \qquad (4.3) \\
v_{i,i} &= 0, \qquad (4.4) \\
\dot{T} &= \kappa \Delta T. \qquad (4.5)
\end{aligned}
$$

The appropriate boundary conditions for the velocity vector field reflect the no-slip assumptions at $z = 0$ and $z = H$,

$$v_i = 0, \quad z = 0, H.$$

The temperature is kept fixed at the vertical boundaries $z = 0, H$, such that

$$T = T_0, \quad z = 0, \quad \text{and} \quad T = T_H, \qquad z = H,$$

with $T_0 > T_H$, the fluid being heated from below.

Then a steady solution for the boundary value problem from above is

$$\bar{v}_i \equiv 0, \quad \bar{T} = -\zeta z + T_0$$

where $\zeta = (T_0 - T_H)H^{-1}$. To study the nonlinear stability of stationary solution we let $(u_i,\, \theta,\, \pi)$ be perturbations to $(\bar{v}_i,\, \bar{T},\, \bar{p})$, where $\bar{p}$ is the steady pressure field found from (4.3). The resulting perturbation equations are non-dimensionalized via similar scaling as in the second grade fluid analysis, and the dimensionless perturbed equations, for $(x, y) \in \mathbb{R}^2$ and $z \in (0, 1)$, are

$$
\begin{aligned}
\dot{u}_i &= -\pi_{,i} + \delta_{i3}R\theta + \{F(z)\Pi^{m/2}a_{ij}\}_{,j} + \{f'(\hat{T})\theta\Pi^{m/2}a_{ij}\}_{,j} \\
&\quad + \frac{1}{\Gamma_1}(a_{ij,t} + u_k a_{ij,k} + a_{im}u_{m,j} + u_{m,i}a_{mj})_{,j} \\
&\quad + \frac{1}{\Gamma_2}(a_{im}a_{mj})_{,j}. \qquad (4.6) \\
u_{i,i} &= 0, \\
Pr\dot{\theta} &= \Delta\theta + Rw
\end{aligned}
$$

where $Ra = R^2$, Pr are the Rayleigh and Prandtl numbers, Γ_1, Γ_2 are absorption numbers. We have written $w = u_3$ and $a_{ij} = u_{i,j} + u_{j,i}$ and the Taylor expansion (2.20) has been also considered.

At the boundaries $z = 0$ and $z = 1$

$$u_i = \theta = 0,$$

with u_i, θ and π having a periodic shape in (x, y).

4.2 Conditional nonlinear stability analysis

We multiply $(4.6)_1$ by u_i and integrate over the period cell V, in order to obtain one of the energy identities. After use of integration by parts and the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2] = R\langle\theta\, w\rangle - \langle F(z)\Pi^{m/2}a_{ij}u_{ij}\rangle - \langle f'(\hat{T})\Pi^{m/2}\theta a_{ij}u_{ij}\rangle. \quad (4.7)$$

In the same manner we start with the equation $(4.6)_3$, multiply by θ and integrate over V, to have:

$$\frac{1}{2}\frac{d}{dt}Pr||\theta||^2 = R\langle\theta\, w\rangle - ||\nabla\theta||^2. \quad (4.8)$$

We now add equations (4.7) and λ(4.8), with λ being a positive parameter. It follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2 + \lambda Pr||\theta||^2] &= R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2\\ &- \langle\frac{F(z)}{2}(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle\\ &- \langle\frac{f'(\hat{T})}{2}(tr\mathbf{A}_1^2)^{m/2}\theta a_{ij}a_{ij}\rangle, \quad (4.9)\end{aligned}$$

where $\Pi^{m/2}$ was replaced by $(tr\mathbf{A}_1^2)^{m/2}$.

If we take into account the natural energy

$$E(t) = \frac{1}{2}[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2 + \lambda Pr||\theta||^2]$$

and denote

$$\begin{aligned}\mathcal{I}' &= R(1+\lambda)\langle\theta\, w\rangle,\\ \mathcal{D}' &= \lambda||\nabla\theta||^2 + \langle\frac{F(z)}{2}(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle,\end{aligned}$$

then identity (4.9) becomes

$$\frac{dE}{dt} = \mathcal{I}' - \mathcal{D}' - \langle \frac{f'(\hat{T})}{2}\theta(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle. \tag{4.10}$$

The natural dissipation term in the energy stability analysis, $\mathcal{D}'$ appears to be not sufficient to control the terms which arise in the energy equation, and moreover the existence of a solution for a maximum problem $\max_{\mathcal{H}} \frac{\mathcal{I}'}{\mathcal{D}'}$ over a suitable functional space, $\mathcal{H}$, is questionable. Hence, we propose to modify (4.1) in a way which is physically consistent, but simplifies the stability analysis. We employ

$$\mathbf{T} = -p\mathbf{I} + \mu(T)[1 + \hat{\gamma}\Pi^{m/2}]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{4.11}$$

where $\mu(T)$ is given by (2.18). This modification is consistent with the theory of Dunn & Fosdick [13], and if we regard $\mu(T)[1 + \hat{\gamma}\Pi^{m/2}]$ as the viscosity then the ability to describe the effects of shear thickening and shear thinning is retained. In our analysis we restrict attention to the case when $m > 0$.

With ω being a non-dimensional form for $\hat{\gamma}$ we non-dimensionalize according to the same scalings, but with ν_0 instead of $\bar{\nu}_0$.

The new perturbed equations lead to the following energy identity:

$$\begin{aligned}\frac{dE}{dt} = \quad & R\,(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \langle\frac{F(z)}{2}[1+\omega(tr\mathbf{A}_1^2)^{m/2}]a_{ij}a_{ij}\rangle \\ - \quad & \langle\frac{f'(\hat{T})}{2}\theta[1+\omega(tr\mathbf{A}_1^2)^{m/2}]a_{ij}a_{ij}\rangle,\end{aligned}$$

or equivalently,

$$\begin{aligned}\frac{dE}{dt} = \quad & R\,(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \langle\frac{F(z)}{2}\,a_{ij}a_{ij}\rangle - \langle\frac{F(z)}{2}\,\omega(tr\mathbf{A}_1^2)^{m/2}\,a_{ij}a_{ij}\rangle \\ - \quad & \langle\frac{f'(\hat{T})}{2}\theta\, a_{ij}a_{ij}\rangle - \langle\frac{f'(\hat{T})}{2}\theta\,\omega(tr\mathbf{A}_1^2)^{m/2}\,a_{ij}a_{ij}\rangle.\end{aligned} \tag{4.12}$$

We define now

$$\begin{aligned}\mathcal{I} &= (1+\lambda)\langle\theta\, w\rangle, \\ \mathcal{D} &= \lambda||\nabla\theta||^2 + \langle\frac{F(z)}{2}a_{ij}a_{ij}\rangle,\end{aligned}$$

and

$$\frac{1}{R_E} = \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}, \tag{4.13}$$

with $\mathcal{H}$ being the same functional space as stated in the previous section.

We require $R < R_E$, such that $a = (R_E - R)/R_E > 0$. Then from equation (4.12) we may derive

$$\begin{aligned}\frac{dE}{dt} \leq & - a\mathcal{D} - \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2} a_{ij}a_{ij}\rangle \\ & - \frac{1}{2}\langle f'(\hat{T})\theta\, a_{ij}a_{ij}\rangle - \frac{\omega}{2}\langle f'(\hat{T})\theta(tr\mathbf{A}_1^2)^{m/2} a_{ij}a_{ij}\rangle, \qquad (4.14)\end{aligned}$$

or if we use the estimation (2.19)

$$\begin{aligned}\frac{dE}{dt} \leq & - a\mathcal{D} - \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2} a_{ij}a_{ij}\rangle \\ & + \frac{M}{2}\langle |\theta|\, a_{ij}a_{ij}\rangle + \frac{M\omega}{2}\langle |\theta|(tr\mathbf{A}_1^2)^{m/2} a_{ij}a_{ij}\rangle. \qquad (4.15)\end{aligned}$$

Again, using Cauchy's inequality on the last two terms of the RHS of (4.15), i.e.

$$\begin{aligned}& \frac{M}{2}\langle |\theta|\, a_{ij}a_{ij}\rangle + \frac{M\omega}{2}\langle |\theta|(tr\mathbf{A}_1^2)^{m/2} a_{ij}a_{ij}\rangle \\ & \leq \frac{M}{2}||\theta||\,||a_{ij}^2|| + \frac{M\omega}{2}||\theta||\,||(tr\mathbf{A}_1^2)^{1+m/2}||,\end{aligned}$$

leads to difficulties in handling the $||(tr\mathbf{A}^2)^{1+m/2}||$ term, with the E or $\mathcal{D}$ forms as above.

To overcome this, we again employ the embedding inequality, (2.28),

$$\sup_V |\theta| \leq c||\Delta\theta||,$$

to write

$$\begin{aligned}& \frac{M}{2}\langle |\theta|\, a_{ij}a_{ij}\rangle + \frac{M\omega}{2}\langle |\theta|(tr\mathbf{A}_1^2)^{m/2} a_{ij}a_{ij}\rangle \\ & \leq M\sup_V |\theta|\, ||\nabla\mathbf{u}||^2 + \frac{M\omega}{2}\sup_V |\theta|\, \langle (tr\mathbf{A}_1^2)^{1+m/2}\rangle \\ & \leq Mc||\Delta\theta||\, ||\nabla\mathbf{u}||^2 + \frac{Mc\omega}{2}||\Delta\theta||\, \langle (tr\mathbf{A}_1^2)^{1+m/2}\rangle. \qquad (4.16)\end{aligned}$$

We note again, similar to the analysis presented in the previous section, that the absence of the $||\Delta\theta||$ term in the energy formula implies the need for a generalised energy analysis. In order to include such a term in the energy definition, we take the Laplacian of $(4.6)_3$ to find

$$Pr\,\Delta\theta_{,t} + Pr\,[\Delta u_i\theta_{,i} + 2u_{i,j}\theta_{,ij} + u_i\Delta\theta_{,i}] = R\Delta w + \Delta^2\theta, \qquad (4.17)$$

and we also conclude that

$$\Delta\theta = 0, \quad \text{on} \quad z = 0, 1.$$

Now we multiply (4.17) by $\Delta\theta$ and integrate over V to obtain

$$\begin{aligned} \frac{1}{2}Pr\frac{d}{dt}||\Delta\theta||^2 = & - \; Pr\,[\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle] \\ & - \; ||\nabla\,\Delta\theta||^2 \;+\; R\langle\Delta w\,\Delta\theta\rangle. \end{aligned} \tag{4.18}$$

The LHS of (4.18) gives the extra term to be added to the natural energy in order to form the generalised one, which is now defined as

$$\mathcal{E}(t) \;=\; E(t) + \frac{\gamma}{2}Pr\,||\Delta\theta||^2, \tag{4.19}$$

where γ is a parameter at our discretion. The energy inequality is constructed by taking the combination $\gamma \times$ (4.18)+(4.15), namely

$$\begin{aligned} \frac{d\mathcal{E}}{dt} \leq & - \; a\mathcal{D} \;-\; \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle \\ & + \; Mc||\Delta\theta\,||\nabla\mathbf{u}||^2 \;+\; \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\ & - \; \gamma Pr\,[\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle] \\ & - \; \gamma||\nabla\,\Delta\theta||^2 \;+\; \gamma R\langle\Delta w\,\Delta\theta\rangle, \end{aligned} \tag{4.20}$$

where (4.16) has also been employed.

Recalling the estimation (3.24)

$$-\gamma Pr\,[\langle\Delta\theta\,\Delta u_i\,\theta_{,i}\rangle + 2\langle\Delta\theta\,u_{i,j}\,\theta_{,ij}\rangle] \;\leq\; 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2,$$

we obtain from (4.20)

$$\begin{aligned} \frac{d\mathcal{E}}{dt} \leq & - \; a\mathcal{D} \;-\; \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle \\ & - \; \gamma||\nabla\,\Delta\theta||^2 \;+\; \gamma R\langle\Delta w\,\Delta\theta\rangle \\ & + \; Mc||\Delta\theta||\,||\nabla\mathbf{u}||^2 \;+\; \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\ & + \; 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2. \end{aligned} \tag{4.21}$$

We integrate by parts and use the arithmetic-geometric mean inequality to find

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} \leq & - a\mathcal{D} - \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle \\
& + \frac{\gamma R}{2\alpha}||\nabla w||^2 + \frac{\gamma R\alpha}{2}||\nabla\,\Delta\theta||^2 - \gamma||\nabla\,\Delta\theta||^2 \\
& + Mc||\Delta\theta||\,||\nabla\mathbf{u}||^2 + \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\
& + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\nabla\,\Delta\theta||^2,
\end{aligned} \tag{4.22}$$

The idea now is to use the $-a\mathcal{D}$ and $-\gamma||\nabla\,\Delta\theta||^2$ terms to dominate the positive terms, hence

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} \leq & - a\mathcal{D} - \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle \\
& + \frac{\gamma R}{2\alpha}||\nabla w||^2 - (-\frac{\gamma R\alpha}{2} + \gamma)||\nabla\,\Delta\theta||^2 \\
& + Mc||\Delta\theta||\,||\nabla\mathbf{u}||^2 + \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\
& + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\,\nabla\,\Delta\theta||^2.
\end{aligned} \tag{4.23}$$

We choose $\alpha = 1/R$, such that $-\frac{\gamma R\alpha}{2} + \gamma = -\frac{\gamma}{2}$ and

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} \leq & - a\mathcal{D} - \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle \\
& + \frac{\gamma R^2}{2}||\nabla w||^2 - \frac{\gamma}{2}||\nabla\,\Delta\theta||^2 \\
& + Mc||\Delta\theta||\,||\nabla\mathbf{u}||^2 + \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\
& + 2\gamma c_1 Pr\,||\nabla\mathbf{u}||\,||\,\nabla\,\Delta\theta||^2.
\end{aligned} \tag{4.24}$$

Next, observe that

$$\frac{\mathcal{D}}{2} = \frac{\lambda}{2}||\nabla\theta||^2 + \frac{1}{4}\langle F(z)\,tr\mathbf{A}_1^2\rangle \geq \frac{1}{4}\langle F(z)\,tr\mathbf{A}_1^2\rangle,$$

and recall $F(z) > N$, for $z \in (0,1)$, with positive N. Thus

$$\frac{\mathcal{D}}{2} \geq \frac{1}{4}\langle F(z)\,tr\mathbf{A}_1^2\rangle \geq \frac{N}{4}\langle tr\mathbf{A}_1^2\rangle = \frac{N}{2}||\nabla\mathbf{u}||^2.$$

Hence

$$\mathcal{D} \geq \frac{\mathcal{D}}{2} + \frac{N}{2}||\nabla\mathbf{u}||^2.$$

Finally, we conclude that

$$-a\mathcal{D} \leq -\frac{a}{2}\mathcal{D} - \frac{a\,N}{2}||\nabla\mathbf{u}||^2, \tag{4.25}$$

and (4.24) may be reduced to

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} \leq & - \frac{a}{2}\mathcal{D} - \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle \\
& - (\frac{a\,N}{2} - \frac{\gamma R^2}{2})||\nabla \mathbf{u}||^2 - \frac{\gamma}{2}||\nabla\,\Delta\theta||^2 \\
& + Mc||\Delta\theta||\,||\nabla \mathbf{u}||^2 + \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\
& + 2\gamma c_1 Pr\,||\nabla \mathbf{u}||\,||\nabla\,\Delta\theta||^2. \qquad (4.26)
\end{aligned}
$$

Let $\gamma = a\,N/R^2$ and (4.26) becomes

$$\frac{d\mathcal{E}}{dt} \leq -\hat{\mathcal{D}} + \hat{\mathcal{N}} \qquad (4.27)$$

where

$$\hat{\mathcal{D}} = \frac{a}{2}\mathcal{D} + \frac{a\,N}{2R^2}||\nabla\,\Delta\theta||^2 + \frac{\omega}{2}\langle F(z)(tr\mathbf{A}_1^2)^{m/2}a_{ij}a_{ij}\rangle, \qquad (4.28)$$

$$
\begin{aligned}
\hat{\mathcal{N}} = & Mc||\Delta\theta||\,||\nabla \mathbf{u}||^2 + \frac{M\,c\,\omega}{2}||\Delta\theta||\,\langle(tr\mathbf{A}_1^2)^{1+m/2}\rangle \\
& + \frac{2a\,Nc_1 Pr}{R^2}\,||\nabla \mathbf{u}||\,||\nabla\,\Delta\theta||^2. \qquad (4.29)
\end{aligned}
$$

The generalised energy has now the right properties to control the nonlinear term $\hat{\mathcal{N}}$, which is bounded as

$$\hat{\mathcal{N}} \leq A\mathcal{E}^{1/2}\hat{\mathcal{D}}, \qquad (4.30)$$

with

$$A = \frac{M\,c\,R}{N\sqrt{2a\,N\,Pr}}(1 + \frac{2}{a}) + 4\sqrt{2}c_1 Pr\sqrt{\Gamma_1}.$$

We employ (4.30) in (4.27) to have

$$\frac{d\mathcal{E}}{dt} \leq -\hat{\mathcal{D}}(1 - A\mathcal{E}^{1/2}), \qquad (4.31)$$

which insures exponentially decay of the generalised energy $\mathcal{E}$, provided that

$$(a)\,R < R_E, \quad (b)\,\mathcal{E}^{1/2}(0) < \frac{1}{A}. \qquad (4.32)$$

Concluding remark. Therefore, nonlinear stability of the steady solution of (4.3)-(4.5) has been established. The result is conditional and based on a generalised energy analysis.

Chapter 5

Nonlinear stability for the dipolar fluid

The problem of convection in a dipolar fluid is studied, when the viscosity is a general function of temperature. A generalised energy approach is not necessary, as the presence of dissipative terms helps us to control the extra nonlinearities which arise when the viscosity varies with temperature. The equations of Bleustein & Green [2] are formulated in a way suitable to describe the convective instability which occurs when a layer of dipolar fluid is heated from below.

We do include here the stability analysis of the dipolar fluid, to outline the advantages of some nonlinearities present in the dipolar stress tensor over those arising in the second grade tensor. It is useful to remark that the nonlinear analysis proceeds with a natural energy, rather than a generalised one, but still the nonlinear stability result is conditional.

This chapter is submitted for publication (Budu [4]).

5.1 The convection equations for a dipolar fluid

In this chapter we study a particular class of generalised fluid whose viscosity varies with temperature, namely a fluid of dipolar type. The theory of a dipolar fluid was introduced by Bleustein & Green [2] and is thought capable of describing a fluid containing long molecules or a suspension of long molecular particles. These writers took account of microstructure effects by including both the gradient of velocity and the second gradient of velocity as constitutive variables, and they also found it necessary to introduce an appropriate stress tensor. Bleustein & Green [2] also solved the problem of Poiseuille flow in a pipe for a dipolar fluid and showed that a flattened velocity profile could be expected. Since then other problems have been successfully solved; Hills [35], solved the problem of slow flow past a sphere for a dipolar fluid, he established a uniqueness theorem in [36], and demonstrated continuous dependence on the data in the improperly posed backward in time problem in [37]. Straughan [60] showed new effects could be predicted in wave motion and investigated nonlinear stability for the constant viscosity Bernard problem in [62] where the micro-length associated with the theory of [2] was shown to have a strong inhibiting effect on thermal convection.

Franchi & Straughan [24] proposed a linear viscosity relation of Palm et al. [49] for the study of the Bénard problem in a dipolar fluid. In the present work, we replace this linear relation by a more general dependence, as in (2.17).

The appropriate equations for thermal convection in a dipolar fluid are introduced now. The model of Bleustein & Green [2] consists of the momentum equation

$$\rho\dot{v}_i = \rho f_i + \sigma_{ji,j}, \tag{5.1}$$

the continuity equation

$$v_{i,i} = 0,$$

and the rate of work equation

$$\rho r - \rho(\dot{A} + \dot{T}S + \dot{S}T) - q_{i,i} + \tau_{ji}D_{ij} + \Sigma_{(ij)k}A_{kji} = 0 \tag{5.2}$$

where v_i, ρ, f_i, σ_{ji}, r, A, T, S, q_i, are, respectively, the velocity, density, macroscopic body force, stress tensor, heat supply, Helmholtz free energy, temperature, entropy, and heat flux. The superposed dot denotes the material derivative. The tensors D_{ij} and A_{ijk} are

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad A_{ijk} = v_{i,jk},$$

and the stress tensor has the form

$$\sigma_{ji} = \tau_{ji} - \Sigma_{(kj)i,k} - \rho F_{ji} + \rho\Gamma_{ji},$$

where τ_{ij} is a symmetric stress

$$\tau_{ij} = -\phi\delta_{ij} + 2\mu D_{ij}.$$

Here ϕ is introduced since v_i is solenoidal, and μ is the dynamic viscosity, of form (2.18). F_{ij} is the microscopic body force and Γ_{ji} is the dipolar inertia whose form, see Green & Naghdi [30], is

$$\Gamma_{ji} = d^2\left[(\dot{v}_i)_{,j} - v_{i,k}v_{k,j} - v_{i,k}v_{j,k} + v_{k,i}v_{k,j}\right]$$

where d^2 (> 0) is the constant inertia coefficient.

In [62] it is argued that since only the symmetric part $\Sigma_{(kj)i}$ of the dipolar stress plays any part in the equations it is reasonable to introduce only this part for any situation which the dipolar fluid will model. We adopt this premise here and then the constitutive equations of Bleustein & Green [2] yield

$$\begin{aligned}
\Sigma_{(kj)i} &= -\psi_k\delta_{ij} - \psi_j\delta_{ik} + h_1\delta_{jk}A_{imm} + h_2(A_{kji} + A_{jki}) + h_3A_{ij,k} + \gamma_d\delta_{jk}T_{,i},\\
q_i &= \kappa T_{,i} + \bar{\alpha}A_{ikk}.
\end{aligned}$$

The function ψ_i arises because v_i is solenoidal and h_i, γ_d, κ, $\bar{\alpha}$ are constants which satisfy inequalities (15.11) of [2]; the only two of which we require here are

$$\kappa \leq 0, \quad h_1 + h_3 \geq 0$$

and we suppose these hold in the strict sense.

We take a infinite fluid layer $z \in (0, H)$, $H > 0$ and $F_{ji} = 0$. The Boussinesq approximation is considered here, that $\rho = \rho_0$ (constant) everywhere except in the body force term in (5.1), for which

$$\rho f_i = -\rho_0 g\delta_{i3}\left[1 - \alpha(T - T_0)\right]$$

where g is gravity, T_0 is a reference temperature, and α is the coefficient of thermal expansion.

By setting $k = -\frac{\kappa}{\rho_0}c$, with $c = T\frac{\partial S}{\partial T}$ (constant), it is shown in [62] that the rate of work equation may be reduced to

$$\dot{T} = k\Delta T.$$

Therefore, if we now define $\hat{\Sigma}_{(kj)i}$ as

$$\hat{\Sigma}_{(kj)i} = h_1\delta_{jk}A_{imm} + h_2(A_{kji} + A_{jki}) + h_3A_{ij,k}$$

the governing equations (5.1)-(5.2) reduce to

$$\begin{aligned}
&\rho_0(1 - d^2\Delta)\,\dot{v}_i + \rho_0 d^2\left\{v_{i,k}v_{k,j} + v_{i,k}v_{j,k} - v_{k,i}v_{k,j}\right\}_{,j} = \\
&\qquad -p_{,i} - \rho_0 g\delta_{i3}\left[1 - \alpha(T - T_0)\right] \\
&\qquad +2\left\{\mu(T)D_{ij}\right\}_{,j} - \hat{\Sigma}_{(kj)i,kj} - \gamma_d\,\Delta T_{,i}, \\
&v_{i,i} = 0, \\
&\dot{T} = k\Delta T,
\end{aligned} \tag{5.3}$$

where $p = \phi - \psi_{i,i}$ acts like a pressure.

No-slip boundary conditions for the velocity are

$$v_i = 0, \quad z = 0, H, \tag{5.4}$$

and further, the temperatures are kept fixed on the vertical boundaries

$$T = T_0, \quad z = 0, \tag{5.5}$$

$$T = T_H, \quad z = H, \tag{5.6}$$

with $T_0 > T_H$, so the fluid is heated from below.

Let $(u_i,\ \theta,\ \pi)$ be perturbations to the steady solution $(\bar{v}_i,\ \bar{T},\ \bar{p})$ of (5.3)-(5.6), where

$$\bar{v}_i \equiv 0, \quad \bar{T} = -\zeta z + T_0 \tag{5.7}$$

with $\zeta = (T_0 - T_H)H^{-1}$, and $\bar{p}$ being the steady pressure field found from $(5.3)_1$.

The perturbed equations are non-dimensionalized via

$$x_i = x_i^* H, \quad u_i = u_i^* U, \quad \pi = \pi^* P, \quad \theta = \theta^* \tilde{T},$$

$$U = \frac{\nu_0}{H}, \quad P = \frac{\nu_0 \, \mu_o}{H^2}, \quad \tilde{T} = U\sqrt{\frac{Pr \, \beta}{\alpha g}}, \quad \mathcal{T} = \frac{H^2}{\nu_0},$$

$$t = t^* \mathcal{T}, \quad Pr = \frac{\nu_0}{k}, \quad R = \sqrt{\frac{\alpha \beta g H^4}{\nu_0 k}}, \quad \Gamma_d = \gamma_d \sqrt{\frac{\beta}{\alpha g k \nu_0 \rho_0^2 H^2}},$$

and

$$\frac{d^2}{H^2} = \delta, \quad \frac{l^2}{H^2} = \epsilon,$$

with l^2 being the non-dimensional form of $(h_1 + h_3)/\mu_0$.

The non-dimensional perturbed equations, for $z \in (0,1)$, become:

$$\begin{aligned}
&(1 - \delta\Delta)\, \dot{u}_i + \delta\{u_{i,k}u_{k,j} + u_{i,k}u_{j,k} - u_{k,i}u_{k,j}\}_{,j} = \\
&\quad - \pi_{,i} + \delta_{i3} R\theta + \{f(\bar{T} + \theta)a_{ij}\}_{,j} \\
&\quad - \frac{1}{\mu_0 H^2}\, \hat{\Sigma}_{(kj)i,kj} - \Gamma_d\, \Delta\theta_{,i}, \\
&u_{i,i} = 0, \\
&Pr\dot{\theta} = \Delta\theta + Rw
\end{aligned} \tag{5.8}$$

where $w = u_3$ and $a_{ij} = u_{i,j} + u_{j,i}$.

Using (2.20), equations $(5.8)_1$ may be written

$$\begin{aligned}
&(1 - \delta\Delta)\, \dot{u}_i + \delta\{u_{i,k}u_{k,j} + u_{i,k}u_{j,k} - u_{k,i}u_{k,j}\}_{,j} = \\
&\quad - \pi_{,i} + \delta_{i3} R\theta + \{F(z)a_{ij}\}_{,j} + \{f'(\hat{T})\theta a_{ij}\}_{,j} \\
&\quad\quad - \frac{1}{\mu_0 H^2}\, \hat{\Sigma}_{(kj)i,kj} - \Gamma_d\, \Delta\theta_{,i},
\end{aligned} \tag{5.9}$$

and, cf. Straughan [62]], on the boundaries

$$u_i = \theta = \Sigma_{(33)i} = 0, \quad z = 0, 1, \tag{5.10}$$

with u_i, θ and π having a periodic shape in (x, y).

5.2 Conditional nonlinear stability analysis

The two separate energy identities are derived by multiplying (5.9) by u_i, $(5.8)_3$ by θ and integrating over the period cell V. After use of integration by parts and the boundary conditions we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||\mathbf{u}||^2 + \delta||\nabla\mathbf{u}||^2) &= R\langle\theta\, w\rangle - \epsilon||\Delta\mathbf{u}||^2 \\ &\quad -\frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle, \qquad (5.11)\\ \frac{Pr}{2}\frac{d}{dt}||\theta||^2 &= R\langle\theta\, w\rangle - ||\nabla\theta||^2. \qquad (5.12)\end{aligned}$$

For $\lambda > 0$ to be chosen later, we may form the energy identity

$$\begin{aligned}\frac{dE}{dt} &= R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \epsilon||\Delta\mathbf{u}||^2 \\ &- \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle, \qquad (5.13)\end{aligned}$$

where

$$E(t) = \frac{1}{2}(||\mathbf{u}||^2 + \delta||\nabla\mathbf{u}||^2 + \lambda Pr||\theta||^2), \qquad (5.14)$$

is the natural energy arising from (5.11)-(5.12). If we take into account the estimation (2.19) for the first derivative of the viscosity, from (5.13) we may see that

$$\begin{aligned}\frac{dE}{dt} &\leq R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \epsilon||\Delta\mathbf{u}||^2 \\ &- \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle + \frac{1}{2}M\langle|\theta|a_{ij}a_{ij}\rangle. \qquad (5.15)\end{aligned}$$

We now define

$$\frac{1}{R_E} = \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}, \qquad (5.16)$$

where $\mathcal{H}$ is the space of functions

$$\begin{aligned}\mathcal{H} &= \{u_i, \theta \,|\, u_i \in W^{2,2}(V), u_{i,i} = 0, u_i = 0 \text{ on z} = 0, 1; \\ &\quad \theta \in W^{1,2}(V), \theta = 0 \text{ at z} = 0, 1\}\end{aligned}$$

u_i, θ satisfy a plane tiling periodic planform in x and y, and

$$\mathcal{I} = (1+\lambda)\langle\theta\, w\rangle,$$
$$\mathcal{D} = \lambda||\nabla\theta||^2 + \epsilon||\Delta\mathbf{u}||^2 + \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle.$$

The corresponding energy inequality is then

$$\frac{dE}{dt} \le R\mathcal{I} - \mathcal{D} + \mathcal{N},$$

where

$$\mathcal{N} = \frac{1}{2}M\langle|\theta|a_{ij}a_{ij}\rangle.$$

Further, we derive

$$\begin{aligned}\frac{dE}{dt} &\le - \mathcal{D}(1 - R\frac{\mathcal{I}}{\mathcal{D}}) + \mathcal{N}\\ &\le - \mathcal{D}(1 - R\max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}) + \mathcal{N}\\ &\le - \mathcal{D}(1 - \frac{R}{R_E}) + \mathcal{N},\end{aligned} \tag{5.17}$$

and take

$$R < R_E,$$

which implies that $(1 - R/R_E) = a > 0$.

We wish now to bound the $\mathcal{N}$ term by the energy, E and by the dissipative term, $\mathcal{D}$. We take

$$|\theta| = (sign\,\theta)\,\theta,$$

with $sign\,\theta$ being -1 or 1, when θ is negative, respectively, positive.

We then proceed as follows

$$\frac{1}{2}M\langle|\theta|a_{ij}a_{ij}\rangle = M\langle(sign\,\theta)\theta u_{i,j}u_{i,j}\rangle + M\langle(sign\,\theta)\theta u_{i,j}u_{j,i}\rangle,$$

or further, integrating by parts,

$$\begin{aligned}\frac{1}{2}M\langle|\theta|a_{ij}a_{ij}\rangle = & -M\langle(sign\,\theta)_{,j}\theta u_{i,j}u_i\rangle - M\langle(sign\,\theta)\theta_{,j}u_{i,j}u_i\rangle\\ & - M\langle(sign\,\theta)\theta\,\Delta u_i\,u_i\rangle - M\langle(sign\,\theta)_{,i}\theta u_{i,j}u_j\rangle\\ & - M\langle(sign\,\theta)\theta_{,i}u_{i,j}u_j\rangle.\end{aligned}$$

The first and the fourth term of the RHS of the last identity are zero. When $\theta \neq 0$ then $sign\,\theta$ is a constant, therefore $(sign\,\theta)_{,j} = (sign\,\theta)_{,i} = 0$. When $\theta = 0$, then the terms are zero. Hence

$$\begin{aligned}\frac{1}{2}M\langle|\theta|a_{ij}a_{ij}\rangle &= -M\langle(sign\,\theta)\theta_{,j}u_{i,j}u_i\rangle - M\langle(sign\,\theta)\theta\,\Delta u_i\,u_i\rangle \\ &\quad - M\langle(sign\,\theta)\theta_{,i}u_{i,j}u_j\rangle. \qquad (5.18)\end{aligned}$$

For the analysis ahead we need the result, (cf. Galdi & Straughan [27], Straughan [64]), that there exists a constant c such that

$$\sup_V |\mathbf{u}| \leq c\,||\Delta\mathbf{u}||. \qquad (5.19)$$

The inequality for the supremum of a function (it is proven to work for θ, as well) is very important in the following analysis and furthermore, throughout the present work. The positive constant c depends on the geometry of the domain.

We now use the fact that the maximum value of $sign\,\theta$ is 1, and employ the inequality (5.19) in (5.18), to have

$$\mathcal{N} \leq 2Mc\,||\nabla\theta||\,||\nabla\mathbf{u}||\,||\Delta\mathbf{u}|| + Mc\,||\theta||\,||\Delta\mathbf{u}||^2.$$

To this end, the cubic terms from above are bounded in terms of $\mathcal{D}\,E^{1/2}$,

$$\begin{aligned}2Mc\,||\nabla\theta||\,||\nabla\mathbf{u}||\,||\Delta\mathbf{u}|| &\leq \frac{M\,c\sqrt{2}}{\sqrt{\lambda\epsilon\delta}}\mathcal{D}E^{1/2} \\ Mc\,||\theta||\,||\Delta\mathbf{u}||^2 &\leq \frac{M\,c\sqrt{2}}{\epsilon\sqrt{\lambda Pr}}\mathcal{D}E^{1/2},\end{aligned}$$

where Poincaré inequality has also been used. The bound for the nonlinear term is then

$$\mathcal{N} \leq Mc\left(\frac{\sqrt{2}}{\sqrt{\delta\,\epsilon\lambda}} + \frac{\sqrt{2}}{\epsilon\sqrt{\lambda Pr}}\right)E^{1/2}\,\mathcal{D}. \qquad (5.20)$$

Thus, suppose (5.20) holds, we derive from (5.17)

$$\frac{dE}{dt} \leq -\,a\mathcal{D} + Mc\left(\frac{\sqrt{2}}{\sqrt{\delta\,\epsilon\lambda}} + \frac{\sqrt{2}}{\epsilon\sqrt{\lambda Pr}}\right)E^{1/2}\,\mathcal{D}.$$

Equivalently we may write

$$\frac{dE}{dt} \leq -\,a\mathcal{D}\,(1 - A\,E^{1/2}),$$

where the positive constant A is given by

$$A = \frac{Mc}{a}\left(\frac{\sqrt{2}}{\sqrt{\delta\,\epsilon\lambda}} + \frac{\sqrt{2}}{\epsilon\sqrt{\lambda Pr}}\right).$$

By a similar procedure as in the previous chapters, we observe that provided

$$(a)\ R < R_E, \qquad (b)\ E^{1/2}(0) < \frac{1}{A}$$

the energy decays to 0 as $t \longrightarrow \infty$.

Concluding remark. What we have established is that provided $R < R_E$ and there exists a threshold for the initial amplitude of the energy, then conditional nonlinear stability is achieved for a general viscosity depending on temperature.

It is of importance to remark that the presence of the term $\epsilon||\Delta\mathbf{u}||^2$ in $\mathcal{D}$ was crucial in the analysis above. Due to this fact, it is possible to control the nonlinear term $\mathcal{N}$ using a natural form for the energy, rather than a generalised one. However, as we have already seen in the previous chapters, for a variety of situations, to establish a nonlinear stability result generalised energies are required.

Chapter 6

Nonlinear stability for the third grade fluid

The thermal convection in a layer of fluid of third grade is investigated, with the viscosity being a general function of temperature. We attack the nonlinear stability analysis and prove that *unconditional nonlinear stability* is achieved using a natural energy approach. This shows that, in some sense, the equations for a fluid of third grade are preferable to those for a fluid of second grade or a dipolar fluid.

This work is submitted for publication (Budu [5]).

6.1 The convection equations for a fluid of third grade

We present below the relevant equations for thermal convection in a layer of fluid of third grade heated from below. The stress tensor relation (2.14) is considered, namely

$$\mathbf{T} = -p\,\mathbf{I} + \mu\,\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta(tr\,\mathbf{A}_1^2)\,\mathbf{A}_1,$$

where $\mathbf{A}_1=[A_{ij}]$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors, defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1$$

with $\mathbf{L}$ being the velocity gradient.

We shall assume the normal stress coefficients α_1, α_2 and the coefficient β are constants, satisfying the restrictions, cf. Fosdick & Rajagopal [20],

$$\alpha_1 \geq 0, \quad \beta \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta}. \tag{6.1}$$

We take here $\alpha_1 > 0$, $\beta > 0$ and impose $(6.1)_3$ with μ replaced by the constant μ_0.

We let the viscosity be a general function of temperature as considered in (2.17), and suppose its first derivative is bounded by a positive constant M,

$$\mu(T) = \rho_0\nu(T) = \rho_0\,\nu_0 f(T), \quad |\nu'(T)| \leq M.$$

We then employ a Boussinesq approximation so that $\rho = \rho_0$ (constant) everywhere except in the body force term.

With these considerations, the equations of motion (2.1)-(2.3) for a fluid of third grade heated from below, contained in the infinite horizontal layer $z \in (0, H)$, are

$$\begin{aligned}
\rho_0 \dot{v}_i &= -p_{,i} - \rho_0 g \delta_{i3}\left[1 - \alpha(T - T_0)\right] + \{\mu(T)A_{ij}\}_{,j} \\
&\quad + \alpha_1(A_{ij,t} + v_k A_{ij,k} + A_{im}L_{m,j} + L_{m,i}A_{mj})_{,j} \\
&\quad + \alpha_2\,(A_{im}A_{mj})_{,j} + \beta\left[(tr\,\mathbf{A}_1^2)A_{ij}\right]_{,j}, && (6.2)\\
v_{i,i} &= 0, && (6.3)\\
\rho_0\dot{\varepsilon} &= T_{ij}L_{ij} - q_{i,i} + \rho_0 r,
\end{aligned}$$

where v_i, ρ, g, ε, r, q_i, are, respectively, the velocity, density, gravity, internal energy, heat supply, and heat flux; g is the gravity, T_0 a reference temperature, and α the coefficient of thermal expansion.

We take here $r = 0$, then we may follow the analysis of Franchi & Straughan [22] and reduce the balance of energy to

$$\dot{T} = \kappa\Delta T, \tag{6.4}$$

with Δ being the Laplacian operator and κ the thermal diffusivity.

No-slip conditions on the boundaries $z = 0, H$ are

$$v_i = 0, \tag{6.5}$$

and the same boundaries are assumed to be held at fixed, constant temperatures

$$T = T_0, \quad z = 0, \quad T = T_H, \quad z = H, \tag{6.6}$$

with $T_0 > T_H$. A steady solution for the boundary value problem (6.2)-(6.6) is then

$$\bar{v}_i \equiv 0, \quad \bar{T} = -\zeta z + T_0 \tag{6.7}$$

where $\zeta = (T_0 - T_H)H^{-1}$.

To study the nonlinear stability of solution (6.7) we let $(u_i,\ \theta,\ \pi)$ be perturbations to $(\bar{v}_i,\ \bar{T},\ \bar{p})$, where $\bar{p}$ is the steady pressure field found from (6.2). We non-dimensionalize the resulting perturbed equations with the scalings

$$\begin{aligned}
&x_i = x_i^* H, \quad u_i = u_i^* U, \quad \pi = \pi^* P, \quad \theta = \theta^* \tilde{T}, \quad t = t^* \mathcal{T},\\
&U = \frac{\nu_0}{H}, \quad P = \frac{U \rho_0 \nu_o}{H}, \quad \tilde{T} = U\sqrt{\frac{Pr\,\zeta}{\alpha g}}, \quad \mathcal{T} = \frac{H^2}{\nu_0},\\
&Pr = \frac{\nu_0}{\kappa}, \quad R = \sqrt{\frac{\alpha \zeta g H^4}{\nu_0 \kappa}}, \quad \Gamma_1 = \frac{\rho_0 H^2}{\alpha_1}, \quad \Gamma_2 = \frac{\rho_0 H^2}{\alpha_2}, \quad B = \frac{\beta \nu_0}{\rho_0 H^4}.
\end{aligned}$$

where $Ra = R^2$, Pr are the Rayleigh and Prandtl numbers, Γ_1, Γ_2 are absorption numbers and B is a non-dimensional form of β.

Omitting all stars, the non-dimensional equations for the evolution of the disturbances, when $z \in (0,1)$, become:

$$\begin{aligned}
\dot{u}_i &= -\pi_{,i} + \delta_{i3} R\theta + \{F(z)a_{ij}\}_{,j} + \{f'(\hat{T})\theta a_{ij}\}_{,j}\\
&\quad + \frac{1}{\Gamma_1}\left(a_{ij,t} + u_k a_{ij,k} + a_{im}u_{m,j} + u_{m,i}a_{mj}\right)_{,j}\\
&\quad + \frac{1}{\Gamma_2}(a_{im}a_{mj})_{,j} + B\left[(tr\,A^2)a_{ij}\right]_{,j}. \qquad (6.8)\\
u_{i,i} &= 0,\\
Pr\,\dot{\theta} &= \Delta\theta + Rw,
\end{aligned}$$

where $w = u_3$, $A = [a_{ij}]$, $a_{ij} = u_{i,j} + u_{j,i}$ and the Taylor expansion (2.20) has been considered.

On the boundaries

$$u_i = \theta = 0, \quad z = 0, 1,$$

with u_i, θ and π having a periodic shape in (x, y).

6.2 Linear instability analysis

To proceed with a linear instability analysis we discard the nonlinear terms in the previous equations. We must, therefore, solve the linearised system:

$$\begin{aligned} u_{i,t} &= -\pi_{,i} + \delta_{i3} R\theta + \{F(z)a_{ij}\}_{,j} + \frac{1}{\Gamma_1} a_{ij,tj}, \\ u_{i,i} &= 0, \\ Pr\,\theta_{,t} &= \Delta\theta + Rw, \end{aligned} \tag{6.10}$$

with the boundary conditions

$$u_i = \theta = 0, \quad z = 0, 1. \tag{6.11}$$

Since (6.10) and (6.11) are linear, we may take

$$u_i(\mathbf{x}, t) = e^{\sigma t} u_i(\mathbf{x}), \quad \theta(\mathbf{x}, t) = e^{\sigma t}\theta(\mathbf{x}), \quad \pi(\mathbf{x}, t) = e^{\sigma t}\pi(\mathbf{x}),$$

and then derive

$$\begin{aligned} \sigma\,[u_i - \frac{1}{\Gamma_1}(u_{i,j} + u_{j,i})_{,j}] &= -\pi_{,i} + \delta_{i3} R\theta + \{F(z)a_{ij}\}_{,j}, \\ u_{i,i} &= 0, \\ \sigma\, Pr\,\theta &= \Delta\theta + Rw. \end{aligned} \tag{6.12}$$

The *Principle of exchange of stability* holds here in the strong sense, namely $\sigma \in \mathbb{R}$. In order to prove the last statement, we multiply $(6.12)_1$ by u_i^*, $(6.12)_3$ by θ^* (u_i^* and θ^* are complex conjugates) and integrate over the periodic cell V to obtain

$$\sigma\,[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}\langle a_{ij}\, u_{i,j}^*\rangle] = R\langle\theta\, w^*\rangle - \langle F(z)\, a_{ij}\, u_{i,j}^*\rangle, \tag{6.13}$$

respectively

$$\sigma\, Pr\, ||\theta||^2 = -||\nabla\theta||^2 + R\langle w\,\theta^*\rangle. \tag{6.14}$$

We now rearrange the term $a_{ij}u_{i,j}^* = (u_{i,j} + u_{j,i})u_{i,j}^*$ as

$$\begin{aligned} a_{ij}u_{i,j}^* &= \frac{1}{2}(a_{ij}\, u_{i,j}^* + a_{ji}\, u_{j,i}^*) = \frac{1}{2}(a_{ij}\, u_{i,j}^* + a_{ij}\, u_{j,i}^*) \\ &= \frac{1}{2}\, a_{ij}\,(u_{i,j}^* + u_{j,i}^*) = \frac{1}{2} a_{ij} a_{ij}^*, \end{aligned} \tag{6.15}$$

and add (6.13) to (6.14) to have

$$\sigma\,[||\mathbf{u}||^2 + Pr\,||\theta||^2 + \frac{1}{2\Gamma_1}\langle\, a_{ij}\, a_{ij}^*\rangle] \;=\; -\frac{1}{2}\langle F(z)\, a_{ij}\, a_{ij}^*\rangle - ||\Delta\theta||^2 + R[\langle\theta\,\omega^*\rangle + \langle\omega\,\theta^*\rangle]. \qquad (6.16)$$

Let $\sigma = \sigma_r + i\,\sigma_i$, with σ_r, $\sigma_i \in \mathbb{R}$; since $\langle\theta\,\omega^*\rangle + \langle\omega\,\theta^*\rangle \in \mathbb{R}$, we take the imaginary part of (6.16) to find that

$$\sigma_i\,[||\mathbf{u}||^2 + Pr\,||\theta||^2 + \frac{1}{2\Gamma_1}\langle a_{ij}\, a_{ij}^*\rangle] = 0. \qquad (6.17)$$

Therefore, $\sigma_i = 0$ and the proof of $\sigma \in \mathbb{R}$ is completed.

As σ is a real number, to find the instability boundary, the lowest value of R^2 in (6.12) for which $\sigma > 0$, we solve (6.12) for the smallest eigenvalue R_1^2 when $\sigma = 0$.

The linear system to be solved is then

$$\begin{aligned} \delta_{i3}R\theta + \{F(z)a_{ij}\}_{,j} &= \pi_{,i}, \\ u_{i,i} &= 0, \\ \Delta\theta + Rw &= 0. \end{aligned} \qquad (6.18)$$

We eliminate the pressure from the equation above by taking the third component of the operation curl curl of equation $(6.18)_1$ and using $(6.18)_2$, obtain

$$-R\,\Delta^*\theta - F''(z)(\Delta w - 2w_{zz}) - 2F'(z)\,\Delta w_{,z} - F(z)\Delta^2 w \;=\; 0, \qquad (6.19)$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The physical assumptions on the fluid layer enable us to adopt a normal mode representation for θ and w of form

$$\theta(x,y,z) = \Theta(z)h(x,y), \quad w(x,y,z) = W(z)h(x,y),$$

with $\Theta(z)$ and $W(z)$ being z-dependent functions and $h(x,y)$ is a planform which tiles the plane (x,y) and satisfies the equation (see eg. Chandrasekhar [8])

$$\Delta^* h = -k^2 h,$$

with k being the wavenumber.

We denote $D = d/dz$ and replace the normal representations of θ and w in $(6.18)_3$ and (6.19), to rewrite the system (6.18) as a one-dimensional eigenvalue problem

$$\begin{aligned} &Rk^2\Theta - F''(z)(D^2+k^2)W - 2F'(z)D(D^2-k^2)W - F(z)(D^2-k^2)^2W = 0, \\ &RW + (D^2-k^2)\Theta = 0, \end{aligned} \tag{6.20}$$

with the boundary conditions for two fixed surfaces given by

$$W = \Theta = DW = 0, \quad z = 0, 1. \tag{6.21}$$

The eigenvalue problem (6.20)-(6.21) is solved by the compound matrix method and numerical results are reported and interpreted in the final section of this chapter. The numerical approach is similar to the one presented in Appendix B, except that here we have 20 compound matrix equations. The first step is to vary R until the final condition is satisfied to some pre-assigned tolerance and find that particular value of R with the secant method. We then find numerically

$$Ra_L = R_L^2 = \min_{k^2} R^2(k^2, F(z)),$$

varying k^2, by using the golden section search algorithm.

As one can easily see from (6.20), the Rayleigh number and the wavenumber are dependent on the viscosity variation, so we shall run the same routine for different values of $F(z)$.

For the function $F(z) = f(\bar{T})$ we have employed the general formula of Tippelskirch and additionally, two formulas as used in Straughan [66], for the viscosity of aniline

$$\nu_1(T) = \frac{0.31482}{1 + 0.48727\times10^{-1}T + 0.87490\times10^{-3}T^2}, \tag{6.22}$$

and, respectively, nitrobenzene

$$\nu_2(T) = \frac{2.6202}{1 + 0.26641\times10^{-1}T + 0.14832\times10^{-4}T^2}. \tag{6.23}$$

It was already emphasised in the introductory part of this thesis that the linear theory only yields a boundary for instability. As $\sigma > 0$ implies instability, we expect to have at least one solution unstable for $Ra > Ra_L$. The linear theory does not yield any information on nonlinear stability; therefore it is possible in general for the solution to become unstable at a value of Ra lower than Ra_L, and in this case subcritical instability is said to occur.

In order to complete the stability analysis for the third grade fluid, we develop next an energy method approach. The nonlinear stability analysis will provide a nonlinear critical Rayleigh number, below which nonlinear stability is assured. For a Rayleigh number between the nonlinear and the linear critical value, subcritical instabilities may still occur. However, one of our main results of the analysis below is that the nonlinear critical Rayleigh number is very close to the linear one, delivering a certain stability result.

6.3 Unconditional nonlinear stability analysis

To proceed with a nonlinear stability analysis, we consider the fully nonlinear equations of (6.8),

$$\begin{aligned}
\dot{u_i} &= -\pi_{,i} + \delta_{i3}R\theta + \{F(z)a_{ij}\}_{,j} + \{f'(\hat{T})\theta a_{ij}\}_{,j} \\
&\quad + \frac{1}{\Gamma_1}\big(a_{ij,t} + u_k a_{ij,k} + a_{im}u_{m,j} + u_{m,i}a_{mj}\big)_{,j} \\
&\quad + \frac{1}{\Gamma_2}(a_{im}a_{mj})_{,j} + B\,[(tr\,A^2)a_{ij}]_{,j}. \\
u_{i,i} &= 0, \\
Pr\,\dot{\theta} &= \Delta\theta + Rw.
\end{aligned}$$

We form the energy identities multiplying $(6.8)_1$ by u_i and integrating over the period cell V. After use of integration by parts and the boundary conditions we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\,[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2] &= R\langle\theta\, w\rangle - \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle \\
&- \frac{1}{2}(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2})\langle tr\,A^3\rangle - \frac{B}{2}\langle |A|^4\rangle. \qquad (6.24)
\end{aligned}$$

Similarly, multiply $(6.8)_3$ by θ and integrate over V, to find:

$$\frac{1}{2}\frac{d}{dt}Pr||\theta||^2 = R\langle\theta\, w\rangle - ||\nabla\theta||^2. \qquad (6.25)$$

Adding (6.24) and λ(6.25), with λ being a positive parameter, the result is

$$\begin{aligned}\frac{dE}{dt} &= R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle \\ &-\frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle - \frac{1}{2}(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2})\langle tr\, A^3\rangle - \frac{B}{2}\langle|A|^4\rangle. \qquad (6.26)\end{aligned}$$

where

$$E(t) = \frac{1}{2}\left[||\mathbf{u}||^2 + \frac{1}{\Gamma_1}||\nabla\mathbf{u}||^2 + \lambda Pr||\theta||^2\right]$$

is the energy in which terms naturally arise from (6.24) and (6.25).

Note. We remark here that we split the analysis in two parts, according to whether $\alpha_1+\alpha_2 = 0$ or $0 < |\alpha_1+\alpha_2| < \sqrt{24\beta\mu_0}$. The third case $|\alpha_1+\alpha_2| = \sqrt{24\beta\mu_0}$ leads to not very useful results.

(i). The case $\alpha_1+\alpha_2 = 0$. Equation (6.26) reduces to

$$\begin{aligned}\frac{dE}{dt} &= R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle \\ &-\frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle - \frac{B}{2}\langle|A|^4\rangle, \qquad (6.27)\end{aligned}$$

or equivalently,

$$\frac{dE}{dt} = \mathcal{I}' - \mathcal{D}' + \mathcal{N}, \qquad (6.28)$$

if we take into account the following notations

$$\begin{aligned}\mathcal{I}' &= R(1+\lambda)\langle\theta\, w\rangle, \\ \mathcal{D}' &= \lambda||\nabla\theta||^2 + \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle, \\ \mathcal{N} &= -\frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle - \frac{B}{2}\langle|A|^4\rangle.\end{aligned}$$

At this point, it is of importance to highlight the role of the β term from the third grade fluid stress tensor. The extra-nonlinearity arising from this term in the corresponding energy equation, namely the $\frac{B}{2}\langle|A|^4\rangle$ term, changes not only the type of the energy necessary to proceed in the stability analysis, but the nonlinear stability result as well.

If we recall the natural energy equation for the second grade fluid, i.e.

$$\begin{aligned}\frac{dE}{dt} &\leq R(1+\lambda)\langle\theta\, w\rangle - \lambda||\nabla\theta||^2 - \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle \\ &- \frac{1}{2}\langle f'(\bar{T})\theta a_{ij}a_{ij}\rangle,\end{aligned}$$

with E being the same as defined above, one can see that the difference between the two energy identities, corresponding to the third grade fluid, respectively, to the second grade fluid, is essentially the nonlinear term corresponding to the β term from (2.14). In the second grade fluid case it was impossible to control the cubic nonlinear term $\langle f'(\bar{T})\theta a_{ij}a_{ij}\rangle$ with the quantities from $\mathcal{I}'$ and $\mathcal{D}'$, therefore a generalised energy analysis was needed in order to deliver a conditional nonlinear stability result. Moreover, in the Navier-Stokes theory, the analysis was more complicated due to the absence of a natural term $||\nabla \mathbf{u}||^2$ in the energy formula, which was added artificially in the equations. As we have remarked in the dipolar stability analysis, extra-nonlinearities from the stress tensor formula have allowed us to proceed with a natural energy formula ending with a conditional nonlinear criteria for stability.

It is first for the third grade fluid analysis that we are able to handle the nonlinear terms by a natural energy analysis and deliver an unconditional boundary for stability. The key to this strong mathematical result is the presence of the extra term $\frac{B}{2}\langle |A|^4\rangle$ which helps us to control the cubic nonlinearity, $\langle f'(\bar{T})\theta a_{ij}a_{ij}\rangle$, in a direct manner.

Essentially, we use (2.19) and the Cauchy inequality to successively have

$$\begin{aligned} \mathcal{N} &= -\frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle - \frac{B}{2}\langle |A|^4\rangle \\ &\le \frac{M}{2}\langle |\theta|\, a_{ij}a_{ij}\rangle - \frac{B}{2}\langle |A|^4\rangle \\ &\le \frac{M}{2}||\theta||\,\langle |A|^4\rangle^{1/2} - \frac{B}{2}\langle |A|^4\rangle. \end{aligned}$$

Further, we employ the arithmetic-geometric mean inequality to obtain

$$\mathcal{N} \le \frac{M}{4\alpha}||\theta||^2 + (\frac{M\alpha}{4} - \frac{B}{2})\langle |A|^4\rangle,$$

for α a positive constant. If we choose now $\alpha = \frac{2B}{M}$, then

$$\mathcal{N} \le \frac{M^2}{8B}||\theta||^2. \tag{6.29}$$

Using this estimation of the nonlinear term $\mathcal{N}$, we rewrite (6.28) as

$$\frac{dE}{dt} \le \mathcal{I} - \mathcal{D}, \tag{6.30}$$

where

$$\begin{aligned} \mathcal{I} &= R(1+\lambda)\langle \theta\, w\rangle + \frac{M^2}{8B}||\theta||^2, \\ \mathcal{D} &= \lambda||\nabla\theta||^2 + \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle. \end{aligned}$$

From (6.30) we derive

$$\frac{dE}{dt} = -\mathcal{D}(1-\frac{\mathcal{I}}{\mathcal{D}}) \leq -\mathcal{D}(1-\max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}), \tag{6.31}$$

where $\mathcal{H}$ is the space of functions

$$\begin{aligned} \mathcal{H} &= \{u_i, \theta \,|\, u_i \in W^{1,2}(V), u_{i,i} = 0, u_i = 0 \text{ at z} = 0, 1; \\ &\qquad \theta \in W^{1,2}(V), \theta = 0 \text{ at z} = 0, 1\} \end{aligned}$$

u_i, θ satisfying a plane tiling periodic planform in x and y.

Upon using Poincaré's inequality in (6.31), the variational theory is then reduced to the maximum problem

$$\max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}} < 1. \tag{6.32}$$

With this assumption we may deduce exponential decay of the energy E. Therefore, (6.32) is an *unconditional nonlinear stability* criterion. We thus have obtained a rigorous nonlinear energy stability result.

The criterion of importance is then (6.32), and everything is reduced to solving the maximum problem. We take (6.32) at critically (equality), and we derive the appropriate Euler-Lagrange equations. Since $\mathcal{H}$ is restricted to those functions that are divergence free, we must add into the maximum problem the constraint $u_{i,i} = 0$ multiplied by a Lagrange multiplier, $p(x)$, $(\int_V p(x)u_{i,i}dx = 0)$.

The Euler-Lagrange equations associated to the maximum problem (6.32) are

$$\begin{aligned} R(1+\lambda)\, w + \frac{M^2}{4B}\theta + 2\lambda\Delta\theta &= 0, \\ R(1+\lambda)\, k_i\theta + 2\{F(z)a_{ij}\}_{,j} &= 2p_{,i} \end{aligned} \tag{6.33}$$

where p is a Lagrange multiplier.

We take the third component of (curl curl)$(6.33)_2$ and decompose into normal modes

$$\theta = \Theta(z)h(x,y), \quad w = W(z)h(x,y),$$

with $\Theta(z)$ and $W(z)$ being z-dependent functions and $h(x,y)$ is a planform which tiles the plane (x,y) and satisfies the equation (see eg. Chandrasekhar [8])

$$\Delta^* h = -k^2 h,$$

with k being the wavenumber.

The equations (6.33) now become

$$\begin{aligned} &R(1+\lambda)W + \frac{M^2}{4B}\Theta + 2\lambda(D^2-k^2)\Theta = 0, \\ &R(1+\lambda)k^2\Theta - 2F''(z)(D^2+k^2)W \\ &\qquad -4F'(z)D(D^2-k^2)W - 2F(z)(D^2-k^2)^2W = 0. \end{aligned} \tag{6.34}$$

System (6.34) is solved subject to the two sets of fixed boundary conditions

$$W = \Theta = DW = 0, \quad z = 0,1. \tag{6.35}$$

This eigenvalue problem is solved numerically by the compound matrix method, with the optimal Rayleigh number of global stability, Ra_E, found by choosing λ such that

$$Ra_E = R_E^2 = \max_{\lambda}\min_{k} R^2(\lambda,k,M,B).$$

The max/min calculations were carried out using the Golden Search algorithm. The numerical results and discussion, with values for Ra_E and critical values of k_E are given in the next section. As Ra_E is dependent on $F(z)$ and B we repeat the algorithm for different values of B and different viscosity variations.

(ii). The case $0 < |\alpha_1+\alpha_2| < \sqrt{24\beta\mu_0}$. We note that

$$2||\nabla \mathbf{u}||^2 + (\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2})\langle tr\, A^3\rangle + B\langle |A|^4\rangle \geq \tag{6.36}$$

$$(1 - \frac{|\alpha_1+\alpha_2|}{2\omega\rho_0H^2\sqrt{6}})\,2\,||\nabla\mathbf{u}||^2 + (\frac{\beta\mu_0}{\rho_0^2H^4} - \frac{\omega|\alpha_1+\alpha_2|}{2\rho_0H^2\sqrt{6}})\langle|A|^4\rangle$$

for $\omega > 0$ at our disposal. Select now

$$\omega = \frac{\sqrt{6}(\frac{\beta\mu_0}{\rho_0H^2} - \rho_0H^2) + \sqrt{6(\frac{\beta\mu_0}{\rho_0H^2} - \rho_0H^2)^2 + (\alpha_1+\alpha_2)^2}}{|\alpha_1+\alpha_2|}$$

and define ε by

$$\varepsilon = 1 - \frac{|\alpha_1 + \alpha_2|}{2\omega\rho_0 H^2\sqrt{6}}. \tag{6.37}$$

It is important for the following analysis to observe that $0 < \varepsilon < 1$, which implies $(1-\varepsilon) > 0$.

We employ (6.36) and (6.37) in (6.26) to conclude that

$$\begin{aligned} \frac{dE}{dt} &\leq R(1+\lambda)\langle\theta\, w\rangle + (1-\varepsilon)||\nabla\mathbf{u}||^2 - \lambda||\nabla\theta||^2 \\ &- \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle - \frac{1}{2}\langle f'(\hat{T})\theta a_{ij}a_{ij}\rangle - \frac{B}{2}\langle|A|^4\rangle. \end{aligned} \tag{6.38}$$

If one uses on the last two nonlinear terms the estimation (6.29), it follows

$$\begin{aligned} \frac{dE}{dt} &\leq R(1+\lambda)\langle\theta\, w\rangle + (1-\varepsilon)||\nabla\mathbf{u}||^2 - \lambda||\nabla\theta||^2 \\ &- \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle + \frac{M^2}{8B}||\theta||^2, \end{aligned} \tag{6.39}$$

which may be reduced to

$$\frac{dE}{dt} \leq \mathcal{I}'' - \mathcal{D}'',$$

where

$$\begin{aligned} \mathcal{I}'' &= R(1+\lambda)\langle\theta\, w\rangle + (1-\varepsilon)||\nabla\mathbf{u}||^2 + \frac{M^2}{8B}||\theta||^2, \\ \mathcal{D}'' &= \lambda||\nabla\theta||^2 + \frac{1}{2}\langle F(z)a_{ij}a_{ij}\rangle. \end{aligned}$$

The criterion for global nonlinear stability is then

$$\max_{\mathcal{H}} \frac{\mathcal{I}''}{\mathcal{D}''} < 1. \tag{6.41}$$

with $\mathcal{H}$ being the space of functions previously defined. The Euler-Lagrange equations for the maximum problem (6.41) are

$$\begin{aligned} &R(1+\lambda)\, w + \frac{M^2}{4B}\theta + 2\lambda\Delta\theta = 0, \\ &R(1+\lambda)\, k_i\theta - 2(1-\varepsilon)\Delta u_i + 2\{F(z)a_{ij}\}_{,j} = -p_{,i} \end{aligned} \tag{6.42}$$

where p is a Lagrange multiplier.

Taking the third component of (curl curl)$(6.42)_2$ and decomposing into normal modes $\theta = \Theta(z)f(x,y)$ and $w = W(z)f(x,y)$, equations (6.42) now transform to

$$\begin{aligned} &R(1+\lambda)\, W + \frac{M^2}{4B}\Theta + 2\lambda(D^2-k^2)\Theta = 0, \\ &R(1+\lambda)\, k^2\Theta - 2F''(z)(D^2+k^2)W - 4F'(z)\, D(D^2-k^2)W \\ &\qquad - 2[F(z) - 1 + \varepsilon]\,(D^2-k^2)^2 W = 0. \end{aligned} \tag{6.43}$$

System (6.43) is solved subject to the fixed boundary conditions (6.35). This is an eigenvalue problem in R which is solved numerically by the compound matrix method, with the optimal Rayleigh number of global stability, Ra_E, found by choosing λ such that

$$Ra_E = R^2 = \max_{\lambda} \min_{k} R^2(\lambda, k, \varepsilon, M, B).$$

6.4 Numerical results and discussion

We have derived *unconditional nonlinear* stability criteria for thermal convection in a layer of fluid of third grade, with the viscosity being a general function of temperature, provided that the first derivative is bounded by a positive constant. It is useful to remark that this strong unconditional nonlinear stability result is due to the extra nonlinear term in the stress relation associated to the third grade fluid.

Although we have proved fully nonlinear stability criteria when ν has a general form, we outline the method for some particular viscosity relations. We include the calculation for the Tippelskirch viscosity formula, completed with some numerical results associated to aniline and nitrobenzene viscosity formulas. The viscosities for these fluids are given by (6.22), respectively, (6.23). The two boundary surfaces are assumed fixed in all cases.

In the present analysis, the two important dimensionless numbers are the critical Rayleigh number, Ra and the wavenumber k. In the following tables we denote the critical values of the Rayleigh numbers in the linear and nonlinear cases as Ra_L, Ra_E respectively. We next compare the behaviour of our numerical results, the critical wavenumbers and the critical Rayleigh numbers, in the linear theory and the energy method.

As both of these numbers are dependent on B and $F(z)$, we take B from very large values to very small ones coupled with different values for the viscosity.

(a) Numerical results for the Tippelskirch viscosity. The fluid viscosity advocated by Tippelskirch [69], relation (2.16), yields for $F(z)$ the following expression

$$F(z) = f(\bar{T}) = \frac{1}{1 + (\epsilon_1 - \delta_1 z) + (\epsilon_2 - \delta_2 z)^2}, \tag{6.44}$$

where

$$\epsilon_1 = A\,T_0, \quad \epsilon_2 = (B)^{1/2}\,T_0, \tag{6.45}$$

and

$$\delta_1 = A\,\Delta T, \quad \delta_2 = (B)^{1/2}\,\Delta T. \tag{6.46}$$

Here C is ν_0, the viscosity evaluated at $T = 0°C$.

Table a.1 gives the linear critical Rayleigh numbers for various values of ν, whereas Tables a.2.(1-4) present the critical wavenumbers k_E^2, and critical Rayleigh numbers Ra_E of energy theory (unconditional nonlinear stability) for different values of ν and B, provided that $\alpha_1 + \alpha_2 = 0$.

For the case $0 < |\alpha_1 + \alpha_2| < \sqrt{24\beta\mu_0}$, the numerical results are similar to those for the $\alpha_1 + \alpha_2 = 0$ case, with respect to the proportion between ε and 1. The proportion is preserved in the critical Rayleigh number and the wavenumber results.

Linear analysis. The eigenvalue problem arising from the variational characterisation of the linear instability boundary, (6.20), suggests that the Rayleigh numbers depend on the variation of ν with the temperature.

The computational output presented below shows that for $\epsilon_1 = \epsilon_2 = \delta_1 = \delta_2 = 0$ the critical Rayleigh number for the linear instability analysis is the same as the classical one for two fixed surfaces.

Nonlinear analysis. As we have already stated, the important parameter in the nonlinear analysis is B, a non-dimensional form of the β coefficient of the extra nonlinearity term in the stress tensor formula for the third grade fluid. Employing a natural energy method, we obtain a critical Rayleigh number for nonlinear stability, which is very close to the linear one for sufficiently large values of B.

It is now worth remarking that the coefficient of Θ from (6.34) which involves B, $M^2/4B$, is the relevant clue to this behaviour. It suggests that for B very large the coefficient tends to zero and the nonlinear results are the same as the linear ones, as long as B does not dominate M^2. As B decreases sufficiently, it will eventually dominate the M^2 term. At this point, the nonlinear results are still close to the linear ones, suggesting that the linear theory provides good predictions for the convection fluid motion. Naturally, as B gets smaller and smaller, the respective coefficient is getting bigger and bigger increasing the difference between the value of the nonlinear critical Rayleigh number and the associated linear result. This is precisely what is found numerically and presented in Tables a.2.(1-4) for various viscosity values

Generally speaking, B can take any positive value, but the numerical calculations reveal some kind of threshold for it. As we may observe from the Tables a.2.(1-4), for each viscosity value, there is a value for B (italic) such that for any values above it the nonlinear critical Rayleigh number is kept close to the linear one, whereas for the values below it the Rayleigh number decreases faster. Moreover, the numerical code used for the calculations provides a minimal admissible value of B, with a corresponding critical nonlinear Rayleigh number. We stress again that this behaviour is predictable from the coefficient of Θ stated above.

The comparison of the nonlinear theory against the linear one is very good, as long as B is above the associated threshold. Roughly speaking, the difference between the two methods is of order $\mathcal{O}(10^{-1})$ for the Rayleigh numbers and of order $\mathcal{O}(10^{-2})$ for wavenumbers.

In order to control the nonlinear term and provide a global nonlinear stability result, we have introduced the coupling parameter λ and employed an energy method to work out the optimal value for this parameter. The results shown in Tables a.2.(1-4) demonstrate that the optimal value for the coupling parameter is close to 1 for high values of B, and significantly greater than 1 as B decreases below the mentioned threshold. For very large values of B, when the situation is technically reduced to the linear case, the coupling parameter is 1.

(b) Numerical results for aniline and nitrobenzene. To give some examples of viscosity fits by the Tippelskirch formula, we may use the viscosity formulas for aniline, (6.22), and for nitrobenzene, (6.23). When $T_0 = 10°C$ and $\Delta T = 2°C$, the numerical calculations for the case of two fixed surfaces are given in Table b.1 and Table b.2. We can observe the influence of B on the Ra_E and k_E^2 results. The tend is the same as for the previous results.

Concluding remark. We have studied convection in a fluid of third grade, with variable viscosity of the general form (2.17). Under these assumptions, we have proceeded with a natural energy method and we have delivered a fully nonlinear stability criteria. The emphasis here is that the third grade fluid possesses a dissipative term which allows us to control the extra nonlinearities which arise when the viscosity varies with temperature. Thus, the final stability result is strong, in the sense that an unconditional nonlinear stability criterion is provided.

ϵ_1	ϵ_2	δ_1	δ_2	Ra_L	k_L
0	*0*	*0*	*0*	*1707.7617*	9.711472
0.1	0.05	0.02	0.01	1563.8847	9.711429
0.2	0.1	0.04	0.02	1437.5102	9.711329
0.1	0.02	0.02	0.004	1566.3392	9.711425
0.2	0.04	0.04	0.008	1445.8091	9.712286
0.1	0.05	0.1	0.05	1626.4514	9.710295
0.2	0.1	0.2	0.1	1552.5363	9.707239
0.1	0.02	0.1	0.02	1627.7049	9.710182
0.2	0.04	0.2	0.04	1556.9162	9.706802
1.0	0.5	0.5	0.25	910.4098	9.709567

Table a.1: Numerical calculation for critical values of the linear critical Rayleigh number Ra_L and the linear critical wavenumber k_L, for various Tippelskirch's viscosity values.

B	Ra_E	k_E^2	λ_c
LINEAR	**1563.8847**	**9.711429**	-
≥ 10	1563.8844	9.711428	1.000000
1	1563.8816	9.711414	1.000004
10^{-1}	1563.8533	9.711292	1.000040
10^{-2}	*1563.5703*	*9.710082*	*1.000402*
10^{-3}	1560.7459	9.697979	1.004024
10^{-4}	1532.9599	9.579758	1.040485
10^{-5}	1295.7372	8.636968	1.425793
5×10^{-6}	1099.6289	7.951929	1.884728
3.5315×10^{-6}	964.74300	7.136395	2.000000

Table a.2.1: Numerical calculation for nonlinear critical values of the Rayleigh number Ra_E and the wavenumber k_E^2 against B, for the Tippelskirch viscosity, with $\epsilon_1 = 0.1$, $\epsilon_2 = 0.05$, $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $M = 0.01777$.

B	Ra_E	k_E^2	λ_c
LINEAR	**1437.5102**	**9.711329**	-
$\geq 10^2$	1437.5101	9.711328	1.000000
10	1437.5092	9.711325	1.000001
1	1437.5009	9.711287	1.000013
10^{-1}	1437.4180	9.710897	1.000128
10^{-2}	*1436.5888*	*9.707032*	*1.001283*
10^{-3}	1428.3401	9.668642	1.012854
10^{-4}	1350.0211	9.312331	1.130971
5×10^{-5}	1271.2145	8.969306	1.266785
1.1265×10^{-5}	886.87478	7.136777	1.999934

Table a.2.2: Numerical calculation for nonlinear critical values of the Rayleigh number Ra_E and the wavenumber k_E^2 against B, for the Tippelskirch viscosity with $\epsilon_1 = 0.2$, $\epsilon_2 = 0.1$, $\delta_1 = 0.04$, $\delta_2 = 0.02$ and $M = 0.03175$.

B	Ra_E	k_E^2	λ_c
LINEAR	**1566.3392**	**9.711423**	-
≥ 10	1566.3389	9.711421	1.000001
1	1566.3362	9.711412	1.000004
10^{-1}	1566.3096	9.711296	1.000038
10^{-2}	*1566.0431*	*9.710156*	*1.000378*
10^{-3}	1563.3822	9.698774	1.003784
10^{-4}	1537.1803	9.597413	1.038061
10^{-5}	1311.4523	8.687900	1.399343
5×10^{-6}	1121.7994	8.019206	1.828920
3.321×10^{-6}	966.3093	7.136605	2.000000

Table a.2.3: Numerical calculation for nonlinear critical values of the Rayleigh numbers Ra_E and the wavenumber k_E^2 against B, for the Tippelskirch viscosity with $\epsilon_1 = 0.1$, $\epsilon_2 = 0.02$, $\delta_1 = 0.02$, $\delta_2 = 0.004$ and $M = 0.01724$.

B	Ra_E	k_E^2	λ_c
LINEAR	**1445.8091**	**9.712286**	-
$\geq 10^2$	1445.8090	9.712286	1.000000
10	1445.8082	9.712282	1.000001
1	1445.8008	9.712248	1.000011
10^{-1}	1445.7263	9.711902	1.000115
10^{-2}	*1444.9817*	*9.708448*	*1.001146*
10^{-3}	1437.5705	9.674142	1.011473
10^{-4}	1366.8147	9.353196	1.116679
3×10^{-5}	1208.6611	8.681052	11.403232
1.4×10^{-5}	1011.3098	7.933856	1.900788

Table a.2.4: Numerical calculation for nonlinear critical values of the Rayleigh number Ra_E and the wavenumber k_E^2 against B, for the Tippelskirch viscosity with $\epsilon_1 = 0.2$, $\epsilon_2 = 0.04$, $\delta_1 = 0.04$, $\delta_2 = 0.008$ and $M = 0.03$.

B	Ra_E	k_E^2	λ_c
LINEAR	**1131.8487**	**9.710549**	-
$\geq 10^2$	1131.8485	9.710548	1.000000
10	1131.8466	9.710538	1.000004
1	1131.8282	9.710428	1.000036
10^{-1}	1131.6433	9.709333	1.000363
10^{-2}	*1129.7977*	*9.698403*	*1.003632*
10^{-3}	1111.6125	9.591409	1.036522
10^{-4}	954.0423	8.720068	1.382604
5×10^{-5}	820.2715	8.062755	1.793616
3.19×10^{-5}	698.5252	7.138520	2.000000

Table b.1: Numerical calculation for nonlinear critical values of the Rayleigh number Ra_E and the wavenumber k_E^2 against B, for aniline, with $T_0 = 10$, $\Delta T = 2$ and $M = 0.05341$.

B	Ra_E	k_E^2	λ_c
LINEAR	**1376.4037**	**9.711203**	-
$\geq 10^2$	1376.4037	9.711203	1.000000
10	1376.4027	9.711198	1.000002
1	1376.3922	9.711143	1.000017
10^{-1}	1376.2874	9.710636	1.000169
10^{-2}	*1375.2400*	*9.705533*	*1.001693*
10^{-3}	1364.8385	9.655016	1.016972
10^{-4}	1267.6745	9.196977	1.173868
5×10^{-5}	1172.8781	8.775278	1.355717
2.5×10^{-5}	1016.9793	8.138056	1.736935
1.4887×10^{-5}	849.8373	7.142488	2.000000

Table b.2: Numerical calculation for nonlinear critical values of the Rayleigh number Ra_E and the wavenumber k_E^2 against B, for nitrobenzene, with $T_0 = 10$, $\Delta T = 2$ and $M = 0.03647$.

PART II. Convection in a porous medium

Scope and plan of this part. Penetrative convection is investigated in a porous medium bounded above by the ocean bed and below by the interface of the thawing permafrost ground.

In Section 1, we describe the physical phenomenon of such convection flow observed off the coast of Alaska and the mathematical model is presented.

Initially, in Section 2, we consider the linear instability analysis which provides us with a linear critical Rayleigh number. If the linear critical Rayleigh number boundary is exceeded, this ensures instability. It does not preclude the possibility of subcritical instabilities. In order to complete the stability analysis, in Section 3 we use the energy method to determine a nonlinear critical Rayleigh number below which convection cannot develop. This critical value is found to be close to that of linear theory, and therefore no subcritical instability may arise. Finally, in Section 4 we present the numerical results and draw some conclusions.

Most of the work from this chapter has essentially appeared in Budu [3].

Chapter 7

Convection in a porous medium

7.1 The mathematical model for thawing subsea permafrost

The physical situation. The study of convective flow of a fluid in a porous medium is a subject driven by the immense variety of applications in which it arises. To mention some, these applications are in biological, environmental, geophysical and industrial contexts.

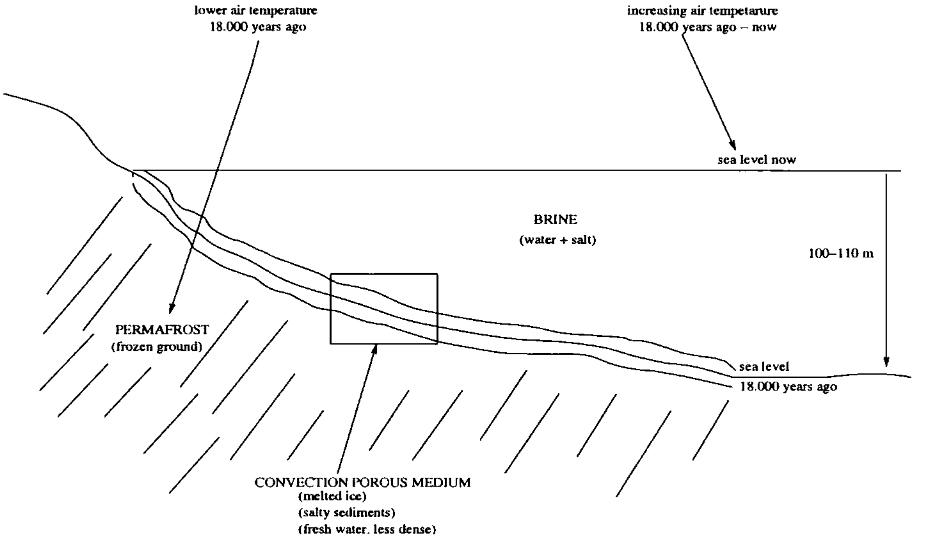

Figure 7.1: The physical situation of the convection in thawing subsea permafrost

Here we present an analysis of the convective motion of brine in a layer of subsea sediments off the coast of Alaska. The physical picture, see Figure 1, arises due to the fact that during the last ice age, approximately 18,000 years ago, the level of the sea was some 100-110 m below its present one. This occurred because water was held in the form of ice in glaciers and massive ice sheets. The air temperature was much colder too with the result that a large layer of permafrost formed in the exposed soil. Even though the air temperature has since warmed and the sea level has risen, the permafrost layer still exists beneath the sea bed. The overlying ocean acts like an insulator to the atmospheric cold. As a result, the permafrost melts from above, with a phase boundary separating the thawed layer and the frozen permafrost. The present permafrost level is therefore a transient one, directly dependent on the relatively warm and salty oceanographic conditions.

The ice in the permafrost is composed of relatively fresh water when compared to the saline water of the sea (brine). This results in a slow, 2-5 cm yr^{-1}, melting of the permafrost layer and has led to the formation of a layer of brine undergoing a convective motion in the soil between the sea bed and the deeper permafrost layer. As the air (and ocean) temperature increased with time, the brine in the thawed permafrost reached unstable conditions and started to circulate. This has led to much research on the topic of convective motion of salt in the layer of sediment beneath the sea bed off the coast off Alaska. The theory behind this motion is that the sea water melts the ice in the permafrost and this releases fresh and less dense water which then rises through the porous layer and a convective motion ensues. This phenomenon has been studied off the coast of Alaska by W. Harrison and co-workers and our analysis is based on a model developed by Harrison & Swift [34].

To incorporate the penetrative convection effect in thawing subsea permafrost, Harrison & Swift [34] and Galdi et al. [26] included the salt effect directly in the equation of state, via a linear dependence of the form

$$\rho = \rho_0(1 + S),$$

where S is the salinity, ρ is the water density and ρ_0 is the reference density. Galdi et al. [26] presented an analysis of linear instability and nonlinear stability. In the nonlinear case they pointed out that if the Rayleigh number is smaller than the critical Rayleigh number, then unconditional nonlinear stability arises. For the case where the nonlinear stability result is conditional upon the existence of some threshold finite amplitude, they calculated a threshold.

Merker et al. [47] and McKay & Straughan [45] studied convection with a nonlinear density depending on temperature, and have shown the nonlinearity can lead to significant effects. Hutter & Straughan [38] include salinity and choose a cubic equation of state for the density in S, from Marchuk & Sarkisyan [44], p.167 and from the UNESCO 1983 paper [72]. Their nonlinear stability result, using the generalised energy method, is a conditional one, which depends upon a threshold for the initial amplitude which they have calculated.

In this work we consider the following UNESCO equation of state for the density, cf. Mellor [46], p. 114,

$$
\begin{aligned}
\rho(T,S) = \; & \rho_0[1 + 6.795021 \times 10^{-5}T - 9.096721 \times 10^{-6}T^2 \\
& +1.001842 \times 10^{-7}T^3 - 1.120259 \times 10^{-9}T^4 \\
& +6.53736 \times 10^{-12}T^5 + 8.246228 \times 10^{-4}S \\
& -4.09054 \times 10^{-6}TS + 7.645003 \times 10^{-8}T^2S \\
& -8.247998 \times 10^{-10}T^3S + 5.388348 \times 10^{-12}T^4S \\
& -5.725561 \times 10^{-6}S^{3/2} + 1.022861 \times 10^{-7}TS^{3/2} \\
& -1.654860 \times 10^{-9}T^2S^{3/2} + 4.832160 \times 10^{-7}S^2]
\end{aligned}
\tag{7.1}
$$

with $\rho_0 = 999.842594$ being a constant, S the salinity corresponding to a solubility of water measured in parts per thousand [‰] and T being the temperature [°C].

We use (7.1) as it retains the accuracy of Hutter & Straughan [38]. But the previous work of Hutter & Straughan [38] has established only conditional nonlinear stability. We prove that using (7.1) we may have a stronger result. Since (7.1) has no cubic term in S as in Hutter & Straughan [38], we may establish a global nonlinear stability result, for all initial data.

The mathematical model. The existence of subsea thawing at temperatures of negative degrees Celsius points to the significance of salt, which in sufficiently permeable sediments percolates through the ocean bed into the thawing layer. Temperature variations may also play a role. However, this effect is much less significant as temperatures are below the anomaly point and density variations are weak. In this model we assume that any convective motion is caused by salt effects rather than temperature.

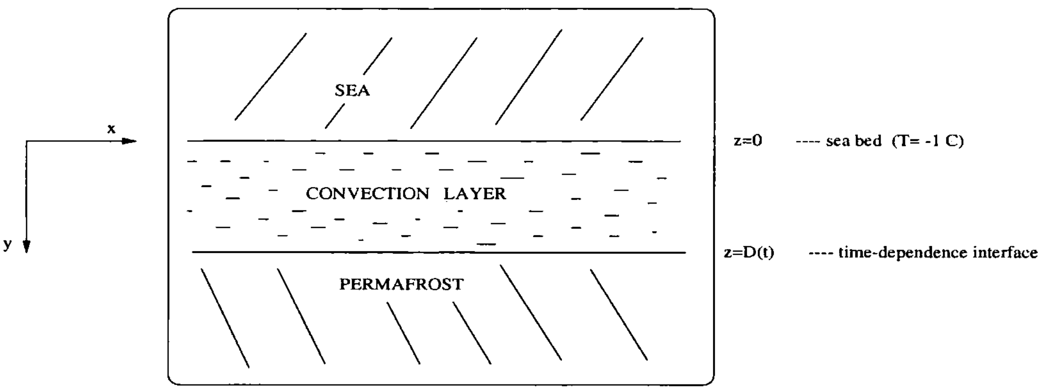

Figure 7.2: Configuration of the thawed layer of salty sediments

The physical situation is as in Figure 7.2. We shall assume for the purpose of calculating the Rayleigh number for the onset of instability, that the permafrost boundary at $z = D(t)$ remains planar. Such an assumption seems reasonable because appreciable boundary movement is on a time scale of years whereas the salt convection is on a time scale of days, and we are analysing the situation before convection commences. The scalings necessary for such a model to be valid are analysed at length in Hutter & Straughan, [39]. The model for the porous layer in which convection may take place is assumed to be governed by Darcy's law, the fluid being incompressible.

Thus, the equations of motion for a spatial domain $\mathbb{R}^2 \times \{z \in (0, D)\}$, are

$$\begin{aligned} u_i &= -\mu\pi_{,i} + k_i g\mu\rho, \\ u_{i,i} &= 0, \\ \frac{\partial S}{\partial t} + u_i S_{,i} &= \kappa\Delta S, \end{aligned} \tag{7.2}$$

where $\mathbf{k} = (0,0,1)$, $u_i, S, \mu, \pi, \rho, \kappa, g$ are velocity, salt concentration, permeability divided by dynamic viscosity, pressure, density, salt diffusivity and gravity. The density is given by (7.1). Standard notation is adopted, the operator Δ is the

Laplacian. Equation $(7.2)_1$ is Darcy's law with a buoyancy term, equation $(7.2)_2$ states that the fluid is incompressible, and the last equation $(7.2)_3$ describes the evolution of the salt concentration field.

Since (7.2) employs Darcy's law, the velocity boundary condition for the porous medium is as follows:

$$u_3 = 0 \quad , \quad z = 0, D.$$

Further, the temperature and the salt concentration on the sea bed may be taken to be constant, namely:

$$T = T_0 \text{ (constant)}, \quad z = 0,$$

$$S = S_0 \text{ (constant)}, \quad z = 0,$$

and for the moving boundary $z = D$, two Stefan conditions must hold for the temperature field and for the salt field, cf. Harrison & Swift [34] and Hutter & Straughan [39],

$$\begin{aligned} L\dot{D} &= -K_1 \frac{\partial T}{\partial z}|_{D^-}, \quad z = D, \\ S(D)\dot{D} &= -\kappa \frac{\partial S}{\partial z}|_{D^-}, \quad z = D. \end{aligned} \tag{7.3}$$

Here L is the coefficient of latent heat per unit volume of the salty layer. The subscript D^- indicates the derivative approaching D from the thawed layer and K_1 is the thermal conductivity above the boundary. In deriving the last two equations, the gradients of temperature and the salt from the permafrost layer are assumed negligible.

Equations (7.3) are combined to eliminate $\dot{D}$. Since the temperature profile in the thawed sediments is nearly linear, the temperature gradient is replaced as follows

$$\frac{\partial T}{\partial z} = \frac{T(D) - T_0}{D}.$$

Following Hutter & Straughan, [38], the temperature profile throughout the layer is then

$$\bar{T} = T_0 - [\frac{T_0 - T(D)}{D}]\, z \tag{7.4}$$

and the density becomes $\rho(\bar{T}, S)$.

To complete the formulation of the model, Harrison [32] points out that the brine concentration and temperature are coupled by the requirement of phase equilibrium at the phase boundary. Phase equilibrium requires

$$S(D) \propto -T(D),$$

where $T(D)$ is the temperature at the phase boundary. With this condition we may write

$$\frac{S(D)}{S_r} = \frac{T(D)}{T_0} \tag{7.5}$$

where S_r is the salinity of water that would begin to freeze at the sea-bed temperature T_0. Here, $T(D) < T_0 < 0$ and so $S(D) > S_r$.

Using (7.5) and eliminating $\dot{D}$ between the Stefan conditions (7.3) for the moving boundary, the final nonlinear boundary condition is

$$\frac{\partial S}{\partial z} = \frac{K_1 T_0}{L\kappa D} S(D) \, [\frac{S(D)}{S_r} - 1], \quad z = D.$$

The problem on a moving region $z \in (0, D(t))$ is hence converted to one in a fixed spatial region $z \in (0, D)$, but with the nonlinear boundary condition given above. So the boundary conditions for $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$ are

$$\begin{aligned} &u_3 = 0, \quad z = 0, D, \\ &S = S_0, \; T = T_0, \quad z = 0, \\ &\frac{\partial S}{\partial z} = \frac{K_1 T_0}{L\kappa D} S(D) \, [\frac{S(D)}{S_r} - 1], \quad z = D. \end{aligned} \tag{7.6}$$

The system (7.2), with the boundary conditions (7.6), admits a stationary (steady) solution

$$\bar{u}_i = 0, \quad \bar{S} = S_0 - \beta_s z, \tag{7.7}$$

where β_s is the positive solution of the equation

$$\beta_s^2 D^2 - \beta_s D(2S_0 - S_r - \kappa L S_r / K_1 T_0) + S_0(S_0 - S_r) = 0, \tag{7.8}$$

and the steady pressure $\bar{p}$ may be calculated from the first equation of (7.2).

An investigation of the stability of the steady solution is performed by introducing the perturbation variables (u_i, s, p) for $(\bar{u}_i, \bar{S}, \bar{p})$.

If we consider now the constants $\rho_0 = 999.842594$ and $c_1 = 8.246228 \times 10^{-4}$, by using the following notations

$$\begin{aligned} f_0(\bar{T}) &= 1 + 6.79502 \times 10^{-5}\bar{T} - 9.09672 \times 10^{-6}\bar{T}^2 + 1.00184 \times 10^{-7}\bar{T}^3 \\ &\quad -1.12025 \times 10^{-9}\bar{T}^4 + 6.53736 \times 10^{-12}\bar{T}^5, \\ f_1(\bar{T}) &= 1 - 4.96049 \times 10^{-3}\bar{T} + 9.27090 \times 10^{-5}\bar{T}^2 - 1.00021 \times 10^{-6}\bar{T}^3 \\ &\quad + 6.53431 \times 10^{-9}\bar{T}^4, \\ f_2(\bar{T}) &= -6.94324 \times 10^{-3} + 1.24039 \times 10^{-4}\bar{T} - 2.00680 \times 10^{-6}\bar{T}^2, \\ c &= 5.85984 \times 10^{-4}, \end{aligned}$$

the density equation (7.1) becomes

$$\rho(\bar{T}, S) = \rho_0[f_0(\bar{T}) + c_1 f_1(\bar{T})S + c_1 f_2(\bar{T})S^{3/2} + c_1 c S^2]. \tag{7.9}$$

To derive the perturbation equations we write (7.9) as

$$\begin{aligned} \rho(\bar{T}, \bar{S} + s) &= \rho_0[f_0(\bar{T}) + c_1 f_1(\bar{T})(\bar{S} + s) \\ &+ c_1 f_2(\bar{T})(\bar{S} + s)^{3/2} + c_1 c(\bar{S} + s)^2]. \end{aligned}$$

It turns out that

$$\begin{aligned} \rho(\bar{T}, \bar{S} + s) - \rho(\bar{T}, \bar{S}) &= \rho_0 c_1\{f_1(\bar{T})s + f_2(\bar{T})[(\bar{S} + s)^{3/2} - \bar{S}^{3/2}] \\ &+ c(2\bar{S}s + s^2)\}. \end{aligned} \tag{7.10}$$

By Taylor series we may write

$$(\bar{S} + s)^{3/2} - \bar{S}^{3/2} = \frac{3}{2}\bar{S}^{1/2}s + \frac{3}{8}\bar{\bar{S}}^{-1/2}s^2, \tag{7.11}$$

where $\bar{\bar{S}}$ is a salinity between $\bar{S}$ and $\bar{S} + s$. We assume $\bar{S} + s > 0$ everywhere. Then using (7.10) in (7.2) we find the perturbation (u_i, s, p) satisfies

$$\begin{aligned} u_i &= -\mu p_{,i} + k_i g \mu \rho_0 c_1[f_1(\bar{T}) + \frac{3}{2}f_2(\bar{T})\bar{S}^{1/2} + 2c\bar{S}]s \\ &+ k_i g \mu \rho_0 c_1[\frac{3}{8}f_2(\bar{T})\bar{\bar{S}}^{-1/2} + c]s^2. \end{aligned} \tag{7.12}$$

Equations (7.12) and a perturbed version of $(7.2)_3$ are now non-dimensionalized via:

$$u_i = Uu_i^*, \quad p = Pp^*, \quad s = \tilde{S}s^*, \quad x_i = Dx_i^*, \quad t = \mathcal{T}t^*,$$

$$U = \kappa/D, \quad P = UD/\mu, \quad \mathcal{T} = D^2/\kappa,$$
$$\tilde{S} = \sqrt{\frac{\beta_s \kappa}{g\mu\rho_0 c_1}}, \quad R = \sqrt{\frac{\beta_s g \mu \rho_0 c_1 D^2}{\kappa}},$$

where $R^2 = Ra$ is the Rayleigh number and the starred variables are the non-dimensional ones. We shall drop all the stars in what follows, but it is understood that the new variables are non-dimensional. The boundary initial value problem governing the evolution of the non-dimensional perturbation field is given by the dimensionless system

$$\begin{aligned} u_i &= -p_{,i} + k_i R g_1(z) s + k_i g_2(z) s^2, \\ u_{i,i} &= 0, \quad z \in (0,1), \\ s_{,t} + u_i s_{,i} &= \Delta s + Rw, \end{aligned} \tag{7.13}$$

where $w = u_3$. On the boundaries,

$$\begin{aligned} &u_3 = 0, \quad z = 0, 1, \\ &s = 0, \quad z = 0, \\ &\frac{\partial s}{\partial z} = -as - bs^2, \quad z = 1, \end{aligned} \tag{7.14}$$

with

$$a = \frac{K_1 |T_0|}{L\kappa}\left(\frac{2S_D}{S_r} - 1\right), \quad b = \frac{K_1 |T_0|}{L\kappa}\frac{\tilde{S}}{S_r}.$$

Here $S_D = \bar{S}(D)$. The estimates of the constants given by Harrison & Osterkamp [33] suggest that a will always be positive.

The functions g_1 and g_2 in (7.13) are given by

$$\begin{aligned} g_1(z) &= 1 + 4.96049 \times 10^{-3}(1+\delta z) + 9.27090 \times 10^{-5}(1+\delta z)^2 \\ &+ 1.00021 \times 10^{-6}(1+\delta z)^3 + 6.53431 \times 10^{-9}(1+\delta z)^4 \\ &- 6.16150 \times 10^{-3}(1-\epsilon z)^{1/2} - 1.10073 \times 10^{-4}(1+\delta z)(1-\epsilon z)^{1/2} \\ &- 1.78085 \times 10^{-6}(1+\delta z)^2(1-\epsilon z)^{1/2} + 4.10188 \times 10^{-4}(1-\epsilon z), \\ g_2(z) &= \beta_s D\{2.6035 \times 10^{-3} - 4.65146 \times 10^{-5}(1+\delta z) \\ &- 7.52515 \times 10^{-7}(1+\delta z)^2]\bar{\bar{S}}^{-1/2} + 5.85984 \times 10^{-4}\} \end{aligned}$$

with

$$\delta = |\frac{T_0 - T(D)}{T_0}|, \quad \epsilon = \frac{S_0 - S_D}{S_0}.$$

Both, δ and ϵ, are varying with respect of the depth variable, D. From Harrison and Swift [34] typical values for δ and ϵ are $\delta = 1.4$, $\epsilon = 2.857 \times 10^{-4}$ respectively. Through we let D to vary in what follows, we take these typical values as an approximation of δ and ϵ for our numerical calculations. As the coefficients of $g_1(z)$ and $g_2(z)$ are small numbers, we expect that this constraint will not affect the accuracy of the final results.

We next develop a linearised analysis for the system (7.13)-(7.14), and so find a critical Rayleigh number for linear convection in the porous medium.

7.2 Linear instability analysis

To investigate the linear instability we neglect the nonlinear terms in (7.13), (7.14). We must, therefore, solve the system of equations:

$$\begin{aligned} u_i &= -p_{,i} + k_i R g_1(z) s, \\ u_{i,i} &= 0, \\ \sigma s &= \Delta s + R w, \end{aligned} \tag{7.15}$$

for $z \in (0,1)$, with boundary conditions being

$$\begin{aligned} & w = 0, \quad z = 0, 1, \\ & s = 0, \quad z = 0, \\ & \frac{\partial s}{\partial z} = -as, \quad z = 1. \end{aligned} \tag{7.16}$$

Here σ represents the growth rate due to introducing a time dependence relation, $e^{\sigma t}$, similar to the third grade fluid linearized analysis. The linearity of (7.15) and (7.16) ensures the validity of this time dependence.

In order to reduce system (7.15) to a one-dimensional eigenvalue problem, we first take the operation curl curl of equation $(7.15)_1$ and using $(7.15)_2$ we obtain

$$-\Delta u_i = R[\delta_{j3}(g_1(z)s)_{,ij} - \delta_{i3}\Delta(g_1(z)s)],$$

with δ_{ij} being the Kronecker Delta symbol, so

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

We consider only the third component of the last equality, for $i = 3$

$$\Delta w = R\, g_1(z)\Delta^* s, \tag{7.17}$$

where

$$\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

We now adopt a normal mode representation for s and w of form

$$s(x, y, z) = S(z)h(x, y),$$

$$w(x, y, z) = W(z)h(x, y),$$

with $S(z)$ and $W(z)$ being z-dependent functions and $h(x, y)$ is a planform which tiles the plane (x, y) and satisfies the equation (see Christopherson [9])

$$\Delta^* h = -k^2 h,$$

with k being the wavenumber.

Considering $D = d/dz$, $z \in (0, 1)$, $x, y \in \mathbb{R}^2$ and replacing the normal mode representations of s and w in $(7.15)_3$ and (7.17), the system is reduced to

$$\begin{aligned} (D^2 - k^2)W + R\, g_1(z)k^2 S &= 0, \\ (D^2 - k^2)S + RW &= \sigma S, \end{aligned} \tag{7.18}$$

where the boundary conditions are given by

$$\begin{aligned} &W = 0, \quad z = 0, 1 \\ &S = 0, \quad z = 0 \\ &DS + aS = 0, \quad z = 1. \end{aligned} \tag{7.19}$$

The eigenvalue problem (7.18)-(7.19) is solved by the compound matrix method (see Appendix B) and numerical results are reported and interpreted in Section 7.4.

7.3 Nonlinear stability analysis

The method of energy is used to develop a nonlinear stability analysis. The idea is to find sufficient conditions such that all disturbances s decay to 0 as $t \longrightarrow \infty$. Different forms of the energy functional deliver different conditions for nonlinear stability to take place. We shall prove that the use of a generalised energy formula leads to a *conditional* nonlinear stability result, only. Therefore, the theory of generalised energy stability is very useful, because we can achieve suitable nonlinear stability thresholds by an appropriate choice of generalised energy. But the strongest result for nonlinear energy analysis is when it can provide *global* stability for all initial data, no matter how large are they going to grow. Due to the equation of state formula used here, we can improve the conditional result above, providing a global one. The result is very strong, as we do not have constraints for the initial data. At this point we must stress that a choice of a weighted energy is the key in providing an unconditional criterion for nonlinear stability.

Moreover, the Rayleigh number provided by this analysis is almost identical with the linear Rayleigh number from the linear theory developed in the previous section. The difference between those two numbers is very small, so the boundaries for linear instability and nonlinear stability are almost the same. In this case virtually no subcritical stability may arise, so the stability analysis for our model is completed.

7.3.1 Conditional nonlinear stability analysis

The boundary initial value problem governing the evolution of the non-dimensional perturbation field is given by the dimensionless system

$$\begin{aligned} u_i &= -p_{,i} + k_i R g_1(z) s + k_i g_2(z) s^2, \\ u_{i,i} &= 0, \quad z \in (0,1), \\ s_{,t} + u_i s_{,i} &= \Delta s + R\omega, \end{aligned} \tag{7.20}$$

where the associated boundary conditions are

$$
\begin{aligned}
&u_3 = 0, \quad z = 0,1 \\
&s = 0, \quad z = 0 \\
&\frac{\partial s}{\partial z} = -as - bs^2, \quad z = 1.
\end{aligned}
\tag{7.21}
$$

We now form the energy identities multiplying $(7.20)_1$ by u_i, $(7.20)_3$ by s and integrating over the period cell V. After use of integration by parts we obtain

$$||\mathbf{u}||^2 = R\langle g_1(z)s\,w\rangle + \langle g_2(z)s^2\,w\rangle, \tag{7.22}$$

$$\frac{1}{2}\frac{d}{dt}||s||^2 = R\langle s\,w\rangle - ||\nabla s||^2 - a\int_\Gamma s^2\,dA - b\int_\Gamma s^3\,dA. \tag{7.23}$$

Adding the equations (7.22) and (7.23) to develop a variational problem, the resulting energy identity is

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||s||^2 &= R\langle s\,w\rangle + R\langle g_1(z)s\,w\rangle - ||\nabla s||^2 - ||\mathbf{u}||^2 \\
&- a\int_\Gamma s^2\,dA - b\int_\Gamma s^3\,dA + \langle g_2(z)s^2\,w\rangle.
\end{aligned}
\tag{7.24}
$$

Setting then

$$
\begin{aligned}
\mathcal{I} &= \langle s\,w\rangle + \langle g_1(z)s\,w\rangle, \\
\mathcal{D} &= ||\nabla s||^2 + ||\mathbf{u}||^2 + a\int_\Gamma s^2\,dA, \\
\mathcal{N} &= -b\int_\Gamma s^3\,dA + \langle g_2(z)s^2\,w\rangle,
\end{aligned}
\tag{7.25}
$$

from (7.24) immediately follows

$$\frac{1}{2}\frac{d}{dt}||s||^2 = R\mathcal{I} - \mathcal{D} + \mathcal{N}. \tag{7.26}$$

The first approach would be to consider an natural energy defined by

$$E_1(t) = \frac{1}{2}||s||^2.$$

But, we note that the natural L^2 energy theory is not sufficient for our analysis, due to the presence of the quadratic s^2 part in the nonlinear term (g_2s^2, w), that is hard to be manipulate. Instead, an L^4 theory is developed, in order to achieve a nonlinear stability result. The idea is to generalise the natural energy formula, by adding an extra term to control the nonlinearity. We stress that due to the nonlinear term $\mathcal{N}$, we expect the nonlinear stability result to be a conditional one, dependent upon a threshold for the initial amplitudes.

Let us consider the energy functional define as

$$E(t) = \frac{1}{2}||s||^2 + \frac{\mu}{4}||s^2||^2, \tag{7.27}$$

where μ is a positive constant that we shall select later.

We now derive the auxiliary equations necessary for our analysis. We start by taking the curl curl of $(7.20)_1$ and then the third component of this equation is multiplied by w and integrated over V, to obtain

$$||\nabla w||^2 = R\langle g_1(z)\nabla^* s\, \nabla^* w\rangle + \langle g_2(z)\nabla^* s^2\, \nabla^* w\rangle, \tag{7.28}$$

where $\nabla^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

We next differentiate E and add the result to the combination $\lambda\times(7.28)$, with λ a positive coupling parameter to be chosen later. The result is

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}||s||^2 + \frac{\mu}{4}\frac{d}{dt}||s^2||^2 - \lambda||\nabla w||^2 \\ &+ \lambda R\langle g_1(z)\nabla^* s\, \nabla^* w\rangle + \lambda\langle g_2(z)\nabla^* s^2\, \nabla^* w\rangle. \end{aligned} \tag{7.29}$$

For the L^4 norm, the second term in the RHS of (7.29), the derivation is

$$\begin{aligned} \frac{\mu}{4}\frac{d}{dt}||s^2||^2 &= \frac{\mu}{4}\int_V \frac{\partial}{\partial t}(s^4)\, dV = \mu\int_V s^3\, s_t\, dV \\ &= \mu\int_V s^3(-u_i s_{,i} + Rw + \Delta s)\, dV \\ &= -\frac{1}{4}\mu\int_V u_i(s^4)_{,i}\, dV + R\mu\langle w\, s^3\rangle + \mu\int_V s^3\Delta s\, dV \\ &= +\frac{1}{4}\mu\int_V u_{i,i}s^4\, dV + \frac{1}{4}\mu\int_\Gamma u_i n_i s^4\, dA + R\mu\langle w\, s^3\rangle + \mu\int_V s^3\Delta s\, dV \\ &= R\mu\langle w\, s^3\rangle + \mu\{-\int_V \nabla s^3\nabla s\, dV + \int_\Gamma s^3 s_z\, dA\} \\ &= R\mu\langle w\, s^3\rangle - \frac{3\mu}{4}\int_V \nabla s^2\nabla s^2\, dV + \mu\int_\Gamma s^3(-as - bs^2)\, dA \\ &= R\mu\langle w\, s^3\rangle - \frac{3\mu}{4}||\nabla s^2|| - a\mu\int_\Gamma s^4\, dA - b\mu\int_\Gamma s^5\, dA. \end{aligned} \tag{7.30}$$

Recalling (7.24) and considering the final result of the above derivation, (7.30), the energy identity (7.29) becomes,

$$\begin{aligned}\frac{dE}{dt} &= R\langle s\,w\rangle + R\langle g_1(z)s\,w\rangle - ||\nabla s||^2 - ||u||^2 \\ &- a\int_\Gamma s^2\,dA - b\int_\Gamma s^3\,dA + \langle g_2(z)s^2\,w\rangle \\ &+ R\mu\langle w\,s^3\rangle - \frac{3\mu}{4}||\nabla s^2|| - a\mu\int_\Gamma s^4\,dA - b\mu\int_\Gamma s^5\,dA \\ &- \lambda||\nabla w||^2 + \lambda R\langle g_1(z)\nabla^* s\,\nabla^* w\rangle + \lambda\langle(g_2(z)\nabla^* s^2\,\nabla^* w)\rangle.\end{aligned}$$

It is convenient to use the notations of $(7.25)_1$-$(7.25)_2$ and rewrite the last identity as

$$\begin{aligned}\frac{dE}{dt} &= -\mathcal{D}(1-R\mathcal{I}) - b\int_\Gamma s^3\,dA + \langle g_2(z)s^2\,w\rangle + R\mu\langle w\,s^3\rangle \\ &- \frac{3\mu}{4}||\nabla s^2|| - a\mu\int_\Gamma s^4\,dA - b\mu\int_\Gamma s^5\,dA \\ &- \lambda||\nabla w||^2 + \lambda R\langle g_1(z)\nabla^* s\,\nabla^* w\rangle + \lambda\langle g_2(z)\nabla^* s^2\,\nabla^* w\rangle.\end{aligned} \tag{7.31}$$

Define now

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}}, \tag{7.32}$$

where $\mathcal{H}$ is the space of admissible functions over which the maximum is sought

$$\begin{aligned}\mathcal{H} &= \{u_i, s\,|\,u_i \in L^2(V), u_{i,i} = 0, u_3 = 0 \ at\ z = 0, 1; \\ &\qquad s \in W^{1,2}(V), s = 0\ at\ z = 0\}\end{aligned}$$

u_i and s satisfying a plane tiling periodic planform in x and y.

We may derive then:

$$\begin{aligned}-\mathcal{D}(1-R\mathcal{I}) &= -R\mathcal{D}(\frac{1}{R} - \frac{\mathcal{I}}{\mathcal{D}}) \\ &\le -R\mathcal{D}(\frac{1}{R} - \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}) = -R\mathcal{D}(\frac{1}{R} - \frac{1}{R_E}).\end{aligned}$$

From (7.31) one may see that

$$\begin{aligned}\frac{dE}{dt} &\le -\mathcal{D}(\frac{R_E - R}{R_E}) - b\int_\Gamma s^3\,dA + \langle g_2(z)s^2\,w\rangle + R\mu\langle w\,s^3\rangle \\ &- \frac{3\mu}{4}||\nabla s^2|| - a\mu\int_\Gamma s^4\,dA - b\mu\int_\Gamma s^5\,dA \\ &- \lambda||\nabla w||^2 + \lambda R\langle g_1(z)\nabla^* s\,\nabla^* w\rangle + \lambda\langle g_2(z)\nabla^* s^2\,\nabla^* w\rangle.\end{aligned} \tag{7.33}$$

We take $(R_E - R)/R_E = m$, and assume that $R < R_E$; hence $m > 0$.

For clarity, we denote every term of the RHS of (7.33) by T_1 to T_{10}. The first term on the right is associated with the L^2 theory, whereas T_2, T_5, T_6, T_7 and T_8 are evidently negative, so we try to dominate the remaining terms by these negative ones. We may do this with the help of the positive parameters μ and λ. The idea is to estimate the nonlinearities by $\mathcal{D}E^q$, with $q > 0$.

Note that the second and the seventh term in (7.33) can be bounded in terms of $D(s)$ and $D(s^2)$ as follows below. First, we handle T_2 by using the Cauchy-Schwarz inequality, a boundary estimate for s^2 (A.0.4) and the Poincaré inequality (A.0.2) for $||s||^2$ and $||s^2||^2$, to get

$$\begin{aligned} T_2 = b\int_\Gamma s^3\, dA &\leq b(\int_\Gamma s^4\, dA)^{1/2}(\int_\Gamma s^2\, dA)^{1/2} \\ &\leq 2\,b\,||s||^{1/2}||s^2||^{1/2}D^{1/4}(s)D^{1/4}(s^2) \\ &\leq \frac{4}{\pi}\,b\,D^{1/2}(s)D^{1/2}(s^2). \end{aligned}$$

We use now the arithmetic geometric mean inequality which gives rise to a positive constant chosen to be m

$$T_2 \leq b\frac{4}{\pi}\,b\,D^{1/2}(s)D^{1/2}(s^2) \leq \frac{16b^2\pi^{-2}}{m}D(s^2) + \frac{m}{4}D(s).$$

For T_7 we use the fact that $s = 0$ on $z = 0$ to split the s^5 term and then employ the Cauchy-Schwarz inequality twice. Furthermore, we observe that $||s_z^2|| \leq D^{1/2}(s^2)$ and use the Sobolev inequality (A.0.3) for $||s^4||$, to obtain

$$\begin{aligned} T_7 = \mu b\int_\Gamma s^5\, dA &= \frac{5}{2}\mu b\int_V s^3 s_z^2\, dV \\ &\leq \frac{5}{2}\mu b(\int_\Gamma s^6\, dA)^{1/2}(\int_\Gamma (s_z^2)^2\, dA)^{1/2} \\ &\leq \frac{5}{2}\mu b\,||s^4||^{1/2}||s^2||^{1/2}||s_z^2|| \\ &\leq \frac{5}{2}\mu b\,||s^4||^{1/2}||s^2||^{1/2}D^{1/2}(s^2) \\ &\leq \frac{5}{2}\mu b\,\gamma D^{1/2}(s^2)||s^2||^{1/2}D^{1/2}(s^2) \\ &\leq \frac{5\sqrt{2}}{2}\mu\, b\,\gamma\,\frac{E^{1/4}}{\mu^{1/4}}D(s^2). \end{aligned}$$

Next, both terms T_9 and T_{10} are handled by the Cauchy-Schwarz inequality applied twice and the arithmetic geometric mean inequality, with the positive constant arising upon calculation chosen to be $\lambda/3$, i.e.

$$\begin{aligned} T_9 = \lambda R\langle g_1(z)\nabla^* s\,\nabla^* w\rangle &\leq \lambda R\, g_1^{max}||\nabla^* s||||\nabla^* w|| \\ &\leq \lambda R\, g_1^{max} D^{1/2}(s) D^{1/2}(w) \\ &\leq \frac{3\lambda}{4} R^2 (g_1^{max})^2 D(s) + \frac{\lambda}{3} D(w), \end{aligned}$$

respectively,

$$\begin{aligned} T_{10} = \lambda\langle g_2(z)\nabla^* s^2\,\nabla^* w\rangle &\leq \lambda g_2^{max}||\nabla^* s^2||||\nabla^* w|| \\ &\leq \lambda g_2^{max} D^{1/2}(s^2) D^{1/2}(w) \\ &\leq \frac{3\lambda}{4} (g_2^{max})^2 D(s^2) + \frac{\lambda}{3} D(w), \end{aligned}$$

where we have used that $||\nabla^* s|| \leq ||\nabla s||$, $||\nabla^* s^2|| \leq ||\nabla s^2||$ and $||\nabla^* w|| \leq ||\nabla w||$.

It remains to estimate the third and the fourth term. We employ the Cauchy inequality, result (A.0.6), the Sobolev inequality and the fact that $D^{1/2}(s) \leq \mathcal{D}^{1/2}$. Thus, for T_3 the result is

$$\begin{aligned} T_3 = \langle g_2(z) s^2\, w\rangle &\leq g_2^{max}||s^2||\,||w|| \\ &\leq g_2^{max}||s^2||^{1/2}\,||s^2||^{1/2}\mathcal{D}^{1/2} \\ &\leq g_2^{max}\gamma D^{1/2}(s)\,||s^2||^{1/2}\mathcal{D}^{1/2} \\ &\leq g_2^{max}\gamma D^{1/2}(s)\,\sqrt{2}\frac{E^{1/4}}{\mu^{1/4}}\mathcal{D}^{1/2} \\ &\leq g_2^{max}\gamma\,\sqrt{2}\frac{E^{1/4}}{\mu^{1/4}}\mathcal{D} = \frac{\gamma\,\sqrt{2} g_2^{max}}{\mu^{1/4}}\,E^{1/4}\,\mathcal{D}. \end{aligned}$$

Finally, for T_4, the Cauchy inequality, the Poincaré inequality, the Sobolev inequality, (A.0.5) and the arithmetic geometric mean inequality lead to

$$\begin{aligned} T_4 = R\mu\langle s^3\, w\rangle &\leq R\mu||s^3||\,||w|| \\ &\leq R\mu||s^4||^{1/2}||s^2||^{1/2}||w|| \\ &\leq R\mu||s^4||^{1/2}||s^2||^{1/2}\,\frac{2}{\pi} D^{1/2}(w) \\ &\leq R\mu\,\gamma D^{1/2}(s^2)\,||s^2||^{1/2}\,\frac{2}{\pi} D^{1/2}(w) \\ &\leq \frac{2R\mu\,\gamma}{\pi} D^{1/2}(s^2)\,D^{1/2}(w)\,\sqrt{2}\frac{E^{1/4}}{\mu^{1/4}} \\ &\leq \frac{6R^2\mu^{3/2}\gamma^2 E^{1/2}}{\lambda\pi^2} D(s^2) + \frac{\lambda}{3} D(w). \end{aligned}$$

We now insert all these estimations in inequality (7.33), to conclude that

$$\begin{aligned}\frac{dE}{dt} &\leq -m\mathcal{D} - \frac{3\mu}{4}D(s^2) - a\mu\int_\Gamma s^4\,dA - \lambda D(w)\\ &+ \frac{16b^2\pi^{-2}}{m}D(s^2) + \frac{m}{4}D(s) + \frac{5\sqrt{2}}{2}\mu^{3/4}b\,\gamma\,E^{1/4}\,D(s^2)\\ &+ \frac{\gamma\sqrt{2}g_2^{max}}{\mu^{1/4}}\,E^{1/4}\,\mathcal{D} + \frac{6R^2\mu^{3/2}\gamma^2}{\lambda\pi^2}\,E^{1/2}\,D(s^2) + \frac{\lambda}{3}D(w)\\ &+ \frac{3\lambda}{4}R^2(g_1^{max})^2D(s) + \frac{\lambda}{3}D(w) + \frac{3\lambda}{4}(g_2^{max})^2D(s^2) + \frac{\lambda}{3}D(w).\end{aligned}$$

The $D(w)$ terms are reduced by suitable choice of the positive constants arising from the arithmetic geometric mean inequalities applied.

Recalling the definition of $\mathcal{D}$, one can observe that $D(s) \leq \mathcal{D}$, and upon grouping the remaining terms in a convenient way it follows

$$\begin{aligned}\frac{dE}{dt} \leq &- [\frac{3m}{4} - \frac{3\lambda}{4}R^2(g_1^{max})^2]\,\mathcal{D} + \frac{\gamma\sqrt{2}g_2^{max}}{\mu^{1/4}}\,E^{1/4}\,\mathcal{D}\\ &- [\frac{3\mu}{4} - \frac{16b^2\pi^{-2}}{m} - \frac{3\lambda}{4}(g_2^{max})^2]D(s^2)\\ &+ [\frac{5\sqrt{2}}{2}\mu^{3/4}b\,\gamma\,E^{1/4} + \frac{6R^2\mu^{3/2}\gamma^2}{\lambda\pi^2}\,E^{1/2}]\,D(s^2).\end{aligned} \tag{7.34}$$

To this end, we choose the positive constants λ and μ such that the coefficients of $\mathcal{D}$, respectively $D(s^2)$, to become $m/4$, respectively $\mu/2$. Consequently,

$$\frac{3\,m}{4} - \frac{3\lambda}{4}R^2(g_1^{max})^2 = \frac{m}{4},$$

and

$$\frac{3\mu}{4} - \frac{16b^2\pi^{-2}}{m} - \frac{3\lambda}{4}(g_2^{max})^2 = \frac{\mu}{2}.$$

The values for λ and μ are founded to be

$$\lambda = \frac{2\,m}{3R^2(g_1^{max})^2} > 0,$$

respectively

$$\mu = \frac{64b^2\pi^{-2}}{m} + \frac{2\,m}{R^2}(\frac{g_2^{max}}{g_1^{max}})^2 > 0.$$

Substituting λ and μ in (7.34), the result is

$$\begin{aligned}\frac{dE}{dt} &\leq -[\frac{m}{4} - \frac{\gamma\sqrt{2}g_2^{max}}{\mu^{1/4}}E^{1/4}]\mathcal{D} \\ &- [\frac{\mu}{2} - \frac{5\sqrt{2}}{2}\mu^{3/4}b\,\gamma\,E^{1/4} - \frac{6R^2\mu^{3/2}\gamma^2}{\lambda\pi^2}E^{1/2}]\,D(s^2)\,.\end{aligned}$$

For an exponential decay of the energy, the requirement is that the new coefficients of $\mathcal{D}$ and $D(s^2)$, those in square brackets, be positive.

We shall impose first that

$$C_{\mathcal{D}} = \frac{m}{4} - \frac{\gamma\sqrt{2}g_2^{max}}{\mu^{1/4}}E(0)^{1/4} > 0 \iff (\frac{m}{4})^4 > \frac{4\gamma^4\,(g_2^{max})^4}{\mu}E(0)$$

hence

$$E(0) < \frac{\mu\,m^4}{2^{10}\,\gamma^{-4}\,(g_2^{max})^{-4}}. \tag{7.35}$$

For the coefficient of $D(s^2)$ the condition is

$$C_{D(s^2)} = \frac{\mu}{2} - \frac{5\sqrt{2}}{2}\mu^{3/4}b\,\gamma\,E(0)^{1/4} - \frac{6R^2\mu^{3/2}\gamma^2}{\lambda\pi^2}E(0)^{1/2} > 0.$$

We split the last condition in two, to have

$$\begin{aligned}&(i) \quad \frac{\mu}{4} - \frac{5\sqrt{2}}{2}\mu^{3/4}b\,\gamma\,E(0)^{1/4} > 0 \iff \frac{\mu^4}{2^8} > \frac{5^4 2^2}{2^4}\mu^3 b^4\,\gamma^4\,E(0), \\ &(ii) \quad \frac{\mu}{4} - \frac{6R^2\mu^{3/2}\gamma^2}{\lambda\pi^2}E(0)^{1/2} > 0 \iff \frac{\mu^2}{2^4} > \frac{6^2R^4\mu^3\gamma^4}{\lambda^2\pi^4}E(0),\end{aligned}$$

which gives us a second and third threshold for the energy, precisely

$$\begin{aligned}&(i) \quad E(0) < \frac{\mu}{2^6 5^4 b^4\,\gamma^4}, \\ &(ii) \quad E(0) < \frac{\lambda^2\pi^4}{2^6\,3^2R^4\gamma^4}.\end{aligned}$$

With these considerations, it easily follows that

$$\frac{dE}{dt} \leq -C_{\mathcal{D}}\,\mathcal{D} - C_{D(s^2)}\,D(s^2) \leq -C_{\mathcal{D}}\,D(s) - C_{D(s^2)}\,D(s^2).$$

We may then use a continuation argument and the Poincaré's inequality on $||s||^2$, respectively $||s^2||^2$ (see A.0.2, ii), to deduce that

$$\frac{dE}{dt} \leq -N\,E(t), \tag{7.36}$$

for a positive constant N. Integrating (7.36) one obtains

$$E(t) \leq e^{-Nt} E(0),$$

showing that the energy decays exponentially to 0 as $t \to \infty$. This provides us with a nonlinear stability criterion.

The important remark is that for this nonlinear result to hold, it is necessary that the next two restrictions be imposed

1. $R < R_E$,

2. $E(0) < A$, where

$$A = \min\left\{\frac{\mu\, m^4}{2^{10}\,\gamma^4\,(g_2^{max})^4}, \frac{\mu}{2^6 5^4 b^4\,\gamma^4}, \frac{\lambda^2\pi^4}{2^6\, 3^2 R^4 \gamma^4}\right\}.$$

The $R < R_E$ condition determines the nonlinear critical Rayleigh number, whereas $E(0) < A$ is a limitation on the size of the initial amplitudes. Nevertheless, we have achieved here a *conditional* nonlinear stability result.

From the definition of E, provided both conditions above hold, we may see that also $||s||$ and $||s^2||$ decay to 0 as $t \to \infty$. Moreover, from (7.22), using the fact that

$$||\mathbf{u}|| \leq R g_1^{max} ||s|| + g_2^{max} ||s^2||,$$

it is obvious that $||\mathbf{u}|| \to 0$ when $t \to \infty$, as well.

Though we do not calculate the critical nonlinear Rayleigh number here, we expect the value of Ra_E to be very close to that of Ra_L as the system corresponding to the nonlinear case is

$$\begin{aligned} 2\,(D^2 - k^2)W + R\,[g_1(z) + 1]\,k^2 S &= 0, \\ 2\,(D^2 - k^2)S + R\,[g_1(z) + 1]\,W &= 0. \end{aligned} \tag{7.37}$$

One may see that, provided the values of $g_1(z)$ are very close to 1, the solutions of the nonlinear system (7.37) are almost the same as the solutions for the (7.18) system of the linearised analysis. In conclusion, the two critical Rayleigh numbers are very close and no subcritical instability will arise provided $R < R_E$ and $E(0) < A$. Therefore, the stability analysis for thawing subsea permafrost model is completed.

7.3.2 Unconditional nonlinear stability analysis

We have first developed a variational energy stability criterion which effectively involved a generalised energy formula and this has led to a conditional nonlinear stability result. We have shown that for Rayleigh numbers which are smaller than the critical Rayleigh number obtained from a nonlinear analysis the solutions are stable, provided the initial perturbation is sufficiently small. We seek in this section a stronger result of nonlinear stability, an *unconditional* one, where the solution is stable no matter how large the initial amplitude is.

To achieve this we choose an appropriate form for the energy -a weighted energy- in order to remove the nonlinear terms that complicate the analysis. The energy functional employs in this case two parameters, to control both nonlinear terms which complicate the nonlinear analysis.

Note. If we proceed directly from (7.11),

$$(\bar{S}+s)^{3/2}-\bar{S}^{3/2}=\frac{3}{2}\bar{S}^{1/2}s+\frac{3}{8}\bar{\bar{S}}^{-1/2}s^2,$$

then we do not know $\bar{\bar{S}}$ and the equivalent term in a nonlinear analysis may become infinite. To overcome this we argue differently and consider only the linear term of the series above, namely

$$(\bar{S}+s)^{3/2}-\bar{S}^{3/2}=\frac{3}{2}\hat{S}^{1/2}s, \tag{7.38}$$

where $\hat{S}$ is a salinity between $\bar{S}$ and $\bar{S}+s$. Using now (7.38) we find that the perturbation (u_i, s, p) satisfies

$$u_i=-\mu p_{,i}+k_i g\mu\rho_0 c_1[f_1(\bar{T})+\frac{3}{2}f_2(\bar{T})\hat{S}^{1/2}+2c\bar{S}]s+k_i g\mu\rho_0 c_1 c\, s^2$$

and the nonlinear system becomes

$$\begin{aligned} u_i &= -p_{,i}+k_iRG_1(z)s+k_iRG_2(z)\hat{S}^{1/2}s+k_iC\,s^2,\\ u_{i,i} &= 0, \\ s_{,t}+u_is_{,i} &= \Delta s+Rw, \end{aligned} \tag{7.39}$$

for $z \in (0,1)$, where $C = c\beta_s D$, and the functions G_1 and G_2 are given by

$$\begin{aligned} G_1(z) &= 1 + 4.96049 \times 10^{-3}(1+\delta z) + 9.27090 \times 10^{-5}(1+\delta z)^2 \\ &+ 1.00021 \times 10^{-6}(1+\delta z)^3 + 6.53431 \times 10^{-9}(1+\delta z)^4 \\ &+ 4.10188 \times 10^{-4}(1-\epsilon z), \\ G_2(z) &= -1.04148 \times 10^{-2} - 1.86059 \times 10^{-4}(1+\delta z) \\ &- 3.01021 \times 10^{-6}(1+\delta z)^2. \end{aligned}$$

We now form the main energy identities. Multiply $(7.39)_1$ by u_i and integrate over the period cell V to obtain

$$||\mathbf{u}||^2 = R\langle G_1(z)s\,w\rangle + R\langle G_2(z)\hat{S}^{1/2}s\,w\rangle + C\langle s^2\,w\rangle. \tag{7.40}$$

From the salt diffusion equation $(7.39)_3$, multiplying by s and integrating over V it follows

$$\frac{1}{2}\frac{d}{dt}||s||^2 = R\langle s\,w\rangle - D(s) - a\int_\Gamma s^2\,dA - b\int_\Gamma s^3\,dA.$$

The result of adding the last two equations is

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||s||^2 &= R\left[\langle s\,w\rangle + \langle G_1(z)s\,w\rangle + \langle G_2(z)\hat{S}^{1/2}s\,w\rangle\right] - D(s) - ||\mathbf{u}||^2 \\ &- a\int_\Gamma s^2\,dA - b\int_\Gamma s^3\,dA + C\langle s^2\,w\rangle. \end{aligned} \tag{7.41}$$

At this point we mention that if we repeat the process involving the use of a generalised energy formula, an L^4 form instead of a natural L^2 one, the nonlinear stability result is still a conditional one. Due to the density formula stated in (7.1), we believe that we can improve the conditional result above, providing a global one.

Crucial in the proof ahead is a result stated in Galdi et al. [26]. They have delivered an unconditional nonlinear stability result for a density formula linear in S. The idea in their work was to remove the cubic term from (7.41), $-b\int_\Gamma s^3\,dA$, by using the following inequality

$$s\,s_z \leq -\alpha s^2, \tag{7.42}$$

which is true for $z = 1$ and

$$\alpha = \frac{K_1|T_0|}{L\kappa}\left(\frac{S_D}{S_r} - 1\right).$$

The equality above follows from the nonlinear boundary condition, in these steps

$$\begin{aligned} s\,s_z &= -a\,s^2 - b\,s^3 = -a\,s^2 - b\,s^3 + \frac{S_D}{\tilde{S}}\,b\,s^2 - \frac{S_D}{\tilde{S}}\,b\,s^2 \\ &= -(a - \frac{S_D}{\tilde{S}}\,b)\,s^2 - b\,s^2(\frac{S_D}{\tilde{S}} + s) = -\alpha\,s^2 - b\,s^2(\frac{S_D}{\tilde{S}} + s) \end{aligned}$$

Taking into account that we have assumed $s + \bar{S} = S > 0$ at $z = D$ in dimensional variables, it turns out that non-dimensionalising gives $\tilde{S}s + S_D = \tilde{S}(s + S_D/\tilde{S}) > 0$ and finally we obtain (7.42).

Using now (7.42) in (7.41), it immediately follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||s||^2 &\leq R\,[\langle s\,w\rangle + \langle G_1(z)s\,w\rangle + \langle G_2(z)\hat{S}^{1/2}s\,w\rangle] - D(s) - ||\mathbf{u}||^2 \\ &- \alpha\int_\Gamma s^2\,dA + C\langle s^2\,w\rangle, \end{aligned} \tag{7.43}$$

or if we take into account the following notations

$$\begin{aligned} \mathcal{I}_\alpha &= \langle s\,w\rangle + \langle G_1(z)s\,w\rangle + \langle G_2(z)\hat{S}^{1/2}s\,w\rangle, \\ \mathcal{D}_\alpha &= D(s) + ||\mathbf{u}||^2 + \alpha\int_\Gamma s^2\,dA, \\ \mathcal{N}_\alpha &= C\langle s^2\,w\rangle, \end{aligned}$$

then inequality (7.43) may be written as

$$\frac{1}{2}\frac{d}{dt}||s||^2 \leq R\mathcal{I}_\alpha - \mathcal{D}_\alpha + \mathcal{N}_\alpha.$$

Although a device such as (7.42) removes the problem for the first nonlinear term, working with the natural energy, the Galdi et al. [26] technique does not work for our analysis because we run into difficulty with the other nonlinear term $C\langle s^2\,w\rangle$ present in $\mathcal{N}_\alpha$.

To overcome the difficulty posed by the nonlinear terms we make use of a *weighted energy*. Payne & Straughan [50] used a weighted energy of form

$$\int_V (\mu - 2z)\,s^2\,dx, \qquad \mu > 2,$$

to remove the s^2 term in their analysis of penetrative convection. But a slightly different approach is considered here because of the cubic term from the nonlinear boundary condition. We take

$$\hat{\mu} = \mu - \lambda z,$$

where $\mu > \lambda > 0$ are positive constants which we choose later. Note that $\hat{\mu} > 0$, as $z \in (0, 1)$.

We now define the weighted energy by

$$E(t) = \frac{1}{2}\langle \hat{\mu} s \, s \rangle. \tag{7.45}$$

The energy identity is formed multiplying $(7.20)_3$ by $\hat{\mu}s$ and integrating over V to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_V \hat{\mu}(z)s^2 \, dV &= -\frac{1}{2}\int_V u_i\hat{\mu}(z)(s^2)_{,i} \, dV + R\int_V w\hat{\mu}(z)s \, dV \\ &+ \int_V \Delta s \, s\hat{\mu}(z) \, dV. \end{aligned} \tag{7.46}$$

For the first term in the RHS of (7.46), the calculations are

$$-\frac{1}{2}\int_V u_i\hat{\mu}(z)(s^2)_{,i} \, dV = \frac{1}{2}\int_V u_{i,i}\hat{\mu}(z)s^2 \, dV + \frac{1}{2}\int_V u_i\hat{\mu}_{,i}s^2 \, dV.$$

Moreover, as $u_{i,i} = 0$ and $\hat{\mu} = \mu - \lambda z$, then

$$-\frac{1}{2}\int_V u_i\hat{\mu}(z)(s^2)_{,i} \, dV = -\frac{\lambda}{2}\int_V ws^2 \, dV. \tag{7.47}$$

The third term in (7.46) is

$$\begin{aligned} \int_V \Delta s \, s\hat{\mu}(z) \, dV &= -\int_V s_{,i}s_{,i}\hat{\mu} \, dV - \int_V s_{,i}s\hat{\mu}_{,i} \, dV + \int_\Gamma s_z s\hat{\mu}(z) \, dA \\ &= -\int_V \hat{\mu}|\nabla s|^2 \, dV + \frac{\lambda}{2}\int_V (s^2)_{,z} \, dV + \int_\Gamma (s \, s_z)(\mu - \lambda z)_{\Gamma(z=1)} \, dA \\ &= -\int_V \hat{\mu}|\nabla s|^2 \, dV + \frac{\lambda}{2}\int_\Gamma s^2 \, dA + \int_\Gamma (s \, s_z)(\mu - \lambda) \, dA, \end{aligned}$$

or employing the (7.42) estimation for the boundary integral, it follows that

$$\begin{aligned} \int_V \Delta s \, s\hat{\mu}(z) \, dV &\leq -\int_V \hat{\mu}|\nabla s|^2 \, dV + \frac{\lambda}{2}\int_\Gamma s^2 \, dA + \int_\Gamma (-\alpha s^2)(\mu - \lambda) \, dA \\ &\leq -\langle \hat{\mu} \, |\nabla s|^2 \rangle - [-\frac{\lambda}{2} + \alpha(\mu - \lambda)]\int_\Gamma s^2 \, dA. \end{aligned} \tag{7.48}$$

Upon using (7.47) and (7.48) in (7.46), we conclude that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\langle \hat{\mu} s \, s \rangle &\leq -\frac{\lambda}{2}\langle s^2 \, w \rangle - \langle \hat{\mu} \, |\nabla s|^2 \rangle \\ &- \; [-\frac{\lambda}{2} + \alpha(\mu - \lambda)]\int_\Gamma s^2 \, dA + R\langle s \, \hat{\mu} w \rangle. \end{aligned} \tag{7.49}$$

Thus, provided (7.40) as well, the final energy inequality is

$$\begin{aligned}\frac{dE}{dt} &\leq R[\langle s\,\hat{\mu}w\rangle + \langle G_1(z)s\,w\rangle + \langle G_2(z)\hat{S}^{1/2}s\,w\rangle] - ||\mathbf{u}||^2 - \langle\hat{\mu}\,|\nabla s|^2\rangle - \\ &- [-\frac{\lambda}{2} + \alpha(\mu-\lambda)]\int_\Gamma s^2\,dA + (C - \frac{\lambda}{2})\langle s^2\,w\rangle. \end{aligned} \tag{7.50}$$

We may now remove the nonlinear term (s^2, w) requiring that its coefficient be zero, hence

$$C - \frac{\lambda}{2} = 0 \iff \lambda = 2\,C.$$

Since we aim for a nonlinear stability result, we require the coefficient of the boundary term, $-\int_\Gamma s^2\,dA$, to be a positive number, B. This implies

$$B = -\frac{\lambda}{2} + \alpha(\mu-\lambda) > 0 \iff \mu > \lambda(\frac{1}{2\alpha} + 1)\ \ (>\lambda).$$

With these considerations, (7.50) becomes

$$\frac{dE}{dt} \leq R\hat{\mathcal{I}} - \hat{\mathcal{D}} + R\hat{\mathcal{T}}, \tag{7.51}$$

where

$$\begin{aligned}\hat{\mathcal{I}} &= \langle s\,\hat{\mu}w\rangle + \langle G_1(z)s\,w\rangle, \\ \hat{\mathcal{D}} &= ||\mathbf{u}||^2 + \langle\hat{\mu}\,|\nabla s|^2\rangle + B\int_\Gamma s^2\,dA, \\ \hat{\mathcal{T}} &= \langle G_2(z)\hat{S}^{1/2}s\,w\rangle.\end{aligned}$$

The $\hat{S}$ term, from $\hat{\mathcal{T}}$, may be bounded above by the maximum salinity on the ocean bed, which is found to be $SA = 0.28856$ (corresponding to a solubility of 35.7 ‰, as is stated in [41]). If we denote

$$M = (SA)^{1/2}\max_{z\in(0,1)}|G_2(z)|,$$

then employing the Cauchy-Schwarz and the arithmetic-geometric mean inequalities, we may deduce the bound for the $\hat{\mathcal{T}}$ term:

$$\hat{\mathcal{T}} = \langle G_2(z)\hat{S}^{1/2}s\,w\rangle \leq M\,||s||\,||w|| \leq \frac{M}{2}(||s||^2 + ||w||^2).$$

Thus, (7.51) is reduced to

$$\frac{dE}{dt} \leq R\mathcal{I} - \mathcal{D}, \tag{7.52}$$

with

$$\mathcal{I} = \langle s\,\hat{\mu} w\rangle + \langle G_1(z) s\, w\rangle + \frac{M}{2}||s||^2 + \frac{M}{2}||w||^2,$$
$$\mathcal{D} = ||\mathbf{u}||^2 + \langle \hat{\mu}\,|\nabla s|^2\rangle + B\int_\Gamma s^2\,dA.$$

From (7.52) we derive

$$\frac{dE}{dt} = -R\mathcal{D}(\frac{1}{R} - \frac{\mathcal{I}}{\mathcal{D}}) \leq -R\mathcal{D}(\frac{1}{R} - \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}),$$

where $\mathcal{H}$ is the space of admissible functions over which the maximum is sought, the same as stated in the previous analysis. We now define

$$\frac{1}{R_W} = \max_{\mathcal{H}}\frac{\mathcal{I}}{\mathcal{D}}, \tag{7.53}$$

then

$$\frac{dE}{dt} \leq -R\mathcal{D}(\frac{1}{R} - \frac{1}{R_W}). \tag{7.54}$$

Providing that

$$R < R_W, \tag{7.55}$$

then $R^{-1} - {R_W}^{-1} = d > 0$, and from (7.54) we obtain

$$\frac{dE}{dt} \leq -d\,R\,\mathcal{D}.$$

With the aid of Poincaré's inequality, there exists a constant ϵ such that $\mathcal{D} \geq \epsilon\,E$, hence

$$\frac{dE}{dt} \leq -d\,\epsilon\,R\,E.$$

Upon integration this yields

$$E(t) \leq e^{-d\epsilon R t}E(0).$$

What we have established is that provided $R < R_W$, then global nonlinear stability is achieved. The criterion of importance is then (7.55), and everything is reduced to solving the maximum problem (7.53).

Remark. We shall prove in what follows the existence of a maximising solution to (7.53).

Further ahead, we need two inequalities

$$\langle \hat{\mu}\, |\nabla s|^2 \rangle \;\leq\; \mu D(s), \tag{7.57}$$

and

$$D(s) \;\leq\; (\frac{1}{\mu - \lambda U})\langle \hat{\mu}\, |\nabla s|^2 \rangle, \tag{7.58}$$

where

$$U \;=\; \max_{z\in(0,1)} \frac{\int_V z|\nabla s|^2 dV}{\int_V |\nabla s|^2 dV}.$$

The proof of the first one is obvious, as $\hat{\mu} = \mu - \lambda\, z$; a proof for the second is now given.

In order to prove (7.58), if we seek an inequality of form

$$D(s) \;\leq\; (ct)\langle \hat{\mu}\, |\nabla s|^2 \rangle$$

then we obtain step by step

$$\begin{aligned} D(s) \;&\leq\; (ct)\langle \hat{\mu}\, |\nabla s|^2 \rangle \;=\; (ct)\mu D(s) - (ct)\lambda \int_V z|\nabla s|^2 dV \\ (1-(ct)\mu)D(s) \;&\leq\; -(ct)\lambda \int_V z|\nabla s|^2 dV \\ \frac{1-(ct)\mu}{(ct)} \;&\leq\; -\lambda \frac{\int_V z|\nabla s|^2 dV}{\int_V |\nabla s|^2 dV}, \quad \text{for any } z \in (0,1) \end{aligned}$$

or

$$\frac{(ct)\mu - 1}{(ct)} \;\geq\; \lambda \frac{\int_V z|\nabla s|^2 dV}{\int_V |\nabla s|^2 dV}, \quad \text{for any } z \in (0,1)$$

So, we can conclude that

$$\frac{(ct)\mu - 1}{(ct)} \;=\; \lambda \max_{z\in(0,1)} \frac{\int_V z|\nabla s|^2 dV}{\int_V |\nabla s|^2 dV}.$$

If we consider

$$U = \max_{z\in(0,1)} \frac{\int_V z|\nabla s|^2 dV}{\int_V |\nabla s|^2 dV}$$

then

$$\frac{(ct)\mu - 1}{(ct)} = \lambda U,$$

and

$$(ct) = \frac{1}{\mu - \lambda U}.$$

For $z \in (0,1)$, then

$$U = \max_{z\in(0,1)} \frac{\int_V z|\nabla s|^2 dV}{\int_V |\nabla s|^2 dV} \in (0,1).$$

Moreover, as $\mu > \lambda$, then $\mu - \lambda U > 0$. Eventually

$$(ct) = \frac{1}{\mu - \lambda U} > 0.$$

To prove existence of the maximising solution to (7.53), we first prove that $\mathcal{H}$

$$\begin{aligned}\mathcal{H} &= \{u_i, s \,|\, u_i \in L^2(V), u_{i,i} = 0, u_3 = 0 \ at \ z = 0, 1; \\ &\quad s \in W^{1,2}(V), s = 0 \ at \ z = 0\}\end{aligned}$$

where u_i and s satisfying a plane tiling periodic planform in x and y, is a Hilbert space with respect of the norm generated by $\mathcal{D}$. As $\mathcal{H}$ is a topological product of two complete spaces, it is therefore itself complete with respect to the norm endowed by $\mathcal{D}$. The completeness of the space appropriate to u follows from, for example, Temam [68], whereas the space appropriate to s is a subspace of $W^{1,2}(V)$ endowed with a norm equivalent to the standard norm, $||s||^2 + D(s)$, due to the fact that in $\mathcal{H}$ we impose only one boundary condition on s.

In order to prove the last statement we first observe that upon using the Poincaré inequality on $||s||^2$ we obtain

$$||s||^2 + D(s) \leq cD(s) + D(s) = (c+1)D(s),$$

for a positive constant c. We recall now (7.58) in the last estimation, thus

$$||s||^2 + D(s) \leq (c+1)(\frac{1}{\mu - \lambda U}) \langle \hat{\mu} |\nabla s|^2 \rangle$$

and moreover,

$$||s||^2 + D(s) \leq \frac{c+1}{\mu - \lambda U} [\langle \hat{\mu} |\nabla s|^2 \rangle + B \int_\Gamma s^2 \, dA],$$

If we denote $C_1 = (c+1)/(\mu - \lambda U) > 0$ then

$$||s||^2 + D(s) \leq C_1[\langle \hat{\mu} |\nabla s|^2 \rangle + B \int_\Gamma s^2 \, dA].$$

On the other hand, using, respectively, inequality (7.58), the boundary estimation for $||s||^2$, (A.0.4), the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality we derive

$$\begin{aligned}
\langle \hat{\mu}\,|\nabla s|^2\rangle \,+\, B\int_\Gamma s^2\,dA \;&\leq\; \mu\, D(s) \,+\, 2B||s||D^{1/2}(s),\\
&\leq\; \mu\,[D(s) \,+\, ||s||^2] \,+\, B\,[D(s) \,+\, ||s||^2],\\
&\leq\; (\mu + B)[D(s) \,+\, ||s||^2].
\end{aligned}$$

Finally, with $C_2 \,=\, \mu + B > 0$

$$[\,\langle \hat{\mu}\,|\nabla s|^2\rangle \,+\, B\int_\Gamma s^2\,dA] \leq C_2[||s||^2 \,+\, D(s)].$$

Thus, $\langle \hat{\mu}\,|\nabla s|^2\rangle + B\int_\Gamma s^2\,dA$ is equivalent to the standard $W^{1,2}(V)$ norm.

It follows from Poincaré's inequality that $\mathcal{I}/\mathcal{D}$ is bounded above by a constant, β. Therefore, there exist a maximising sequence $\{\mathbf{u}_n, s_n\}$ such that

$$lim\,[\mathcal{I}(w_n, s_n)/\mathcal{D}(\mathbf{u}_n, s_n)] \;=\; \beta.$$

This sequence may be chosen so that $\mathcal{D}(\mathbf{u}_n, s_n) \,=\, 1$ and hence from this we deduce the existence of a subsequence, which will be denoted again by $\{\mathbf{u}_n, s_n\}$, such that

$$u_n \longrightarrow u_0,\;\text{ weakly in}\,(L^2(V))^3, \tag{7.59}$$

$$\begin{aligned}
s_n \;&\longrightarrow\; s_0,\;\text{ weakly in}\,W^{1,2}(V),\\
s_n \;&\longrightarrow\; s_0,\;\text{ strongly in}\,L^2(V),
\end{aligned} \tag{7.60}$$

for some $(\mathbf{u}_0, s_0) \in \mathcal{H}$. Therefore

$$\begin{aligned}
|\mathcal{I}(w_n, s_n) - \mathcal{I}(w_0, s_0)| \;&\leq\; |\langle G(z)\,s_0\,(w_n - w_0)\rangle| \,+\, |\langle G(z)\,w_n\,(s_n - s_0)\rangle|\\
&+\; \frac{M}{2}|\langle s_n^2 - s_0^2\rangle| \,+\, \frac{M}{2}|\langle w_n^2 - w_0^2\rangle|
\end{aligned}$$

with $G(z) \,=\, G_1(z) + \hat{\mu}(z)$. Let G_m be the maximum of $G(z)$ when $z \in (0,1)$ and further we derive

$$\begin{aligned}
|\mathcal{I}(w_n, s_n) - \mathcal{I}(w_0, s_0)| \;&\leq\; \langle G(z)\,s_0\,(w_n - w_0)\rangle| \,+\, G_m\,||w_n||\,||s_n - s_0||\\
&+\; \frac{M}{2}|\langle s_n^2 - s_0^2\rangle| \,+\, \frac{M}{2}|\langle w_n^2 - w_0^2\rangle|
\end{aligned}$$

The first and the last terms converge to zero thanks to (7.59), whereas the middle ones tend to zero as well, due to (7.60). Hence, $\mathcal{I}(w_0, s_0) \;=\; \beta$, and by a simple argument one also shows that $\mathcal{D}(\mathbf{u}_0, s_0) = 1$. The existence of a maximising solution to (7.53) is therefore established.

Since $\mathcal{H}$ is restricted to those functions that are divergence free, we must add into the maximum problem the constraint $u_{i,i} = 0$ multiplied by a Lagrange multiplier, $p(x)$, $(\int_V p(x)u_{i,i}dx = 0)$.

The Euler Lagrange equations for the maximum problem (7.53) are,

$$\begin{aligned} R_W\, G(z,\mu,\lambda)\, w + R_W\, M\, s + 2\hat{\mu}\,\Delta s &= 2\lambda\, s_z, \\ R_W\, G(z,\mu,\lambda)\, s\, k_i + R_W\, M\, w\, k_i + p_{,i} &= 2\, u_i \end{aligned} \tag{7.61}$$

where p is a Lagrange multiplier and

$$G(z,\mu,\lambda) = G_1(z) + \hat{\mu}(z) = G_1(z) + \mu - \lambda\, z.$$

We take the third component of (curl curl)(7.61)$_2$ and decompose into normal modes

$$s = S(z)h(x,y), \quad w = W(z)h(x,y),$$

and equations (7.61) now become

$$\begin{aligned} R_W\, G(z,\mu,\lambda)\, W + R_W\, M\, S + 2\hat{\mu}\,(D^2 - k^2)S &= 2\lambda\, DS, \\ R_W\, G(z,\mu,\lambda)\, Sk^2 + R_W\, M\, Wk^2 + 2(D^2 - k^2)W &= 0, \end{aligned} \tag{7.62}$$

where $D = d/dz$ and k is the wavenumber. System (7.62) is solved subject to the boundary conditions

$$\begin{aligned} W(0) &= S(0) = 0 \\ W(1) &= (\mu - \lambda)DS(1) + B\, S(1) = 0 \end{aligned}$$

This eigenvalue problem is solved numerically by the compound matrix method, with the optimal Rayleigh number of global stability, Ra_W, found by choosing μ such that

$$Ra_W = R_W^2 = \max_{\mu} \min_{k} R^2(\mu, k, B).$$

The max/min calculations were carried out using the Golden Section Search algorithm. The numerical results and discussion, with values for Ra_W and critical values of k_W are given in Section 5. As Ra_W is dependent on B we repeat the algorithm for different values of parameter B, from 0 to 10^8.

7.4 Numerical results and discussion

In the context of thawing subsea permafrost, the two important dimensionless numbers are the critical Rayleigh number, Ra and the wavenumber k. In the following tables we denote the critical values of the Rayleigh numbers in the linear and nonlinear case as Ra_L, Ra_W respectively. We next compare the behaviour of our numerical results, the critical wavenumbers and the critical Rayleigh numbers, in the linear theory and the energy method.

As a and B vary in terms of the depth D, we can find the suitable convection values of Ra which then determines the values of D at which brine convection develops. Table 1 presents the values for the critical Rayleigh number and the critical wavenumber for different values of a. Figure 3 and Figure 4 plot the critical Rayleigh number Ra_L and the wavenumber k_L against a, with a varying from 0 to 100. Table 2 presents the critical wavenumbers k_L, k_W, and critical Rayleigh numbers Ra_L, Ra_W of linear and energy theory (unconditional nonlinear stability) for different values of a, respectively B.

a	Ra_L	k_L^2		a	Ra_L	k_L^2
0	27.0083	5.41038		50	38.6317	9.67879
1	30.1729	6.75016		70	38.8325	9.73196
2	31.9786	7.46922		90	38.9474	9.76197
3	33.1728	7.92148		10^2	38.9882	9.77255
5	34.6725	8.45874		10^3	39.3301	9.85976
10	36.4326	9.03958		10^4	39.3655	9.86862
15	37.2309	9.28334		10^5	39.3690	9.86951
20	37.6875	9.41688		10^6	39.3694	9.86960
30	38.1906	9.55885		10^7	39.3694	9.86961
40	38.4619	9.63314		10^8	39.3694	9.86961

Table 1: Numerical calculations for critical values of the linear critical Rayleigh number Ra_L and the linear critical wavenumber k_L against a, with $\delta = 1.4$ and $\epsilon = 2.857 \times 10^{-4}$

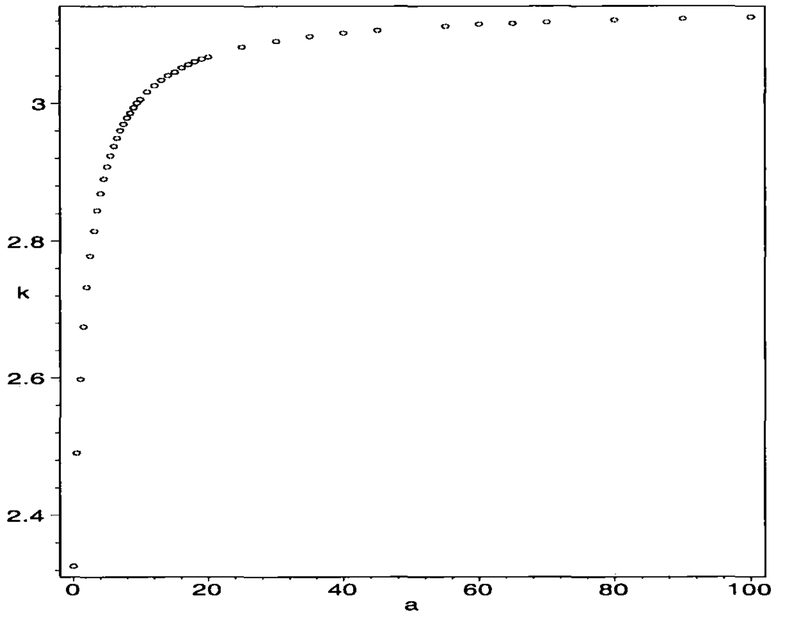

Figure 7.3: Critical values of the linear wavenumber k_L against radiation constant $a \in (0, 100)$, with $\delta = 1.4$ and $\epsilon = 2.857 \times 10^{-4}$

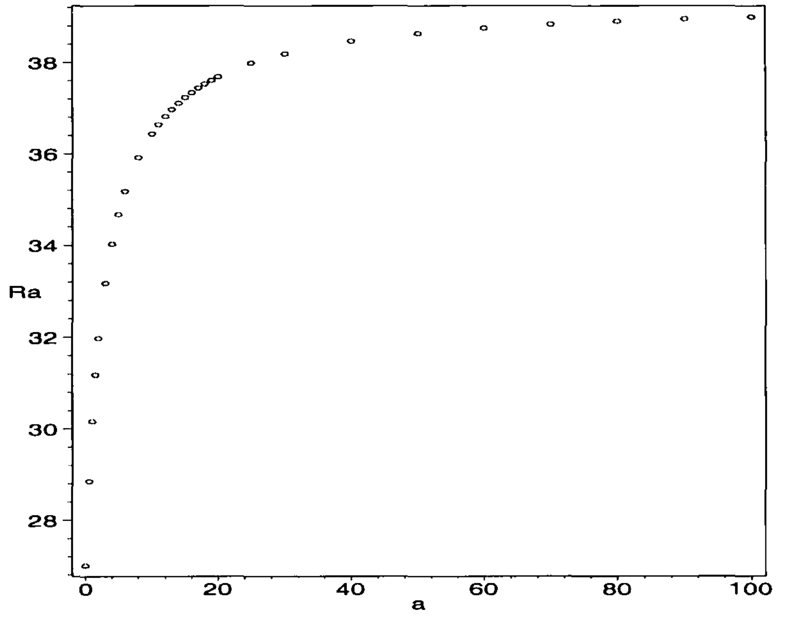

Figure 7.4: Critical values of the linear Rayleigh number Ra_L against radiation constant $a \in (0, 100)$, with $\delta = 1.4$ and $\epsilon = 2.857 \times 10^{-4}$

a	Ra_L	k_L^2	B	Ra_W	k_W^2	μ_c
0	27.0083	5.41038	0	26.838044	5.41036	1.009673
1	30.1729	6.75016	1	30.142396	6.89454	0.8602851
5	34.6725	8.45874	5	34.647985	8.59806	0.8637591
10	36.4326	9.03958	10	36.307591	9.10897	0.9008954
20	37.6875	9.41688	20	37.489249	9.44243	0.9384531
30	38.1906	9.55885	30	37.968707	9.57156	0.9566910
50	38.6317	9.67879	50	38.394222	9.68348	0.9745264
70	38.8326	9.73196	70	38.590031	9.73421	0.9833017
90	38.9474	9.76197	90	38.702707	9.76318	0.9885242
100	38.9882	9.77255	100	38.742836	9.77346	0.9904157
200	39.1756	9.82067	200	38.928097	9.82070	0.9993694
210	39.1847	9.82298	210	38.937117	9.82299	0.9998147
220	39.1929	9.82509	220	38.945333	9.82507	1.000221
230	39.2005	9.82701	230	38.952848	9.82698	1.000594
240	39.2074	9.82977	240	38.959748	9.82873	1.000937
250	39.2138	9.83039	250	38.966106	9.83034	1.001253
400	39.2717	9.84504	400	39.023744	9.84493	1.004139
600	39.3041	9.85321	600	39.056095	9.85311	1.005776
800	39.3203	9.85730	800	39.072360	9.85721	1.006604
10^3	39.3301	9.85976	10^3	39.082149	9.85968	1.007103
10^4	39.3655	9.86862	10^4	39.117569	9.86861	1.008920
10^5	39.3690	9.86951	10^5	39.121127	9.86951	1.009103
10^6	39.3694	9.86960	10^6	39.121483	9.86960	1.009121
10^7	39.3694	9.86961	10^7	39.121519	9.86960	1.009123
10^8	39.3694	9.86961	10^8	39.121522	9.86961	1.009123

Table 2: Numerical calculations for critical values of the Rayleigh numbers Ra_L, Ra_W and the wavenumbers k_L^2, k_W^2 against a, B respectively, with $\delta = 1.4$, $\epsilon = 2.857 \times 10^{-4}$, $\mu_{lower} = 2.331636 \times 10^{-4}$ and $\mu_{upper} = 2$. The μ_c denote the optimal coupling parameter μ.

Linear analysis. From (7.15) and (7.16) is evident that the Rayleigh number will depend on the wavenumber and the parameter a from the boundary condition $(7.16)_3$, so $Ra = Ra(k, a)$. Therefore, this parameter a is important in our analysis. Physically, a may be interpreted as the strength of how much salt is released from the permafrost and flows into the thawed layer across the phase change interface.

Table 1 presents the critical Rayleigh numbers and wavenumbers for different values of this constant a. All these values are calculated employing $\sigma = 0$ and a minimisation technique over the wavenumbers. As we can observe from this table, both the Rayleigh number and the wavenumber, are increasing quickly for small values of a and slowly as values of a increase.

Figures 7.3-7.4 complete the analysis for the linear theory plotting the critical Rayleigh number against the release rate constant a, respectively, the critical wavenumber against the release rate constant a. The graphs reflect the above observation. As we can see, the Rayleigh numbers graph is going asymptotically to 39.3694 as a is growing, whereas the wavenumber graph has the same behaviour and is asymptotic to 3.141593, close to the value of π.

Nonlinear analysis. Using a nonlinear energy stability method, we obtained a critical Rayleigh number for the nonlinear stability, which is very close to that of linear theory. In order to control the nonlinear term and provide a global nonlinear stability result, we have introduced two coupling parameters, λ and μ, employing a weighted energy. As it turns out from Section 4, the λ parameter is fixed, leaving us to work out only the coupling parameter μ, for which we find an optimal value, μ_c. The results shown in Table 2 demonstrate that the optimal value for the coupling parameter is close to 1.

The comparison of nonlinear theory against the linear one is very good. The data presented in Table 2 shows that the linear theory provides predictions very close to those of the energy method. Generally speaking, the difference between both methods is of 10^{-1} order for the Rayleigh number and of 10^{-1}-10^{-2} order for the wavenumber. These results suggest that the linear theory has captured the physics of the onset of convection in the porous medium.

As

$$B = -\frac{\lambda}{2} + \alpha(\mu - \lambda)$$

and

$$a = \alpha + \frac{K_1|T_0|}{L\kappa}\frac{S_D}{S_r},$$

where $\lambda = 2.331636 \times 10^{-4}$, we can observe that the difference between B and a is more or less the value of $K_1|T_0|S_D/L\kappa S_r$. For a fixed depth, $D = 20$ m, the experimental results of Harrison and Swift, [34], state that the value of this quantity is varying between 29 and 290, for different values of κ. This dependence of B and a is reflected roughly in Table 2, as for values of a and B between 1 and 210, the results for the wavenumber values are not those expected. However, as the values are greater than 220, the situation is stabilised and it reflects what we have stated above about the comparison of nonlinear theory against the linear one.

Concluding remark. For the thawing subsea permafrost model values are obtained for the critical Rayleigh number, for both linear and nonlinear stability. From the mathematical point of view the analysis reduces to studying convection in a porous medium with a nonlinear boundary condition.

The critical nonlinear value is found to be close to that of linear theory, and therefore no subcritical instability may arise. We must stress that due to the energy formula which one may use for the nonlinear stability analysis, the nonlinear stability result may be a conditional one, dependent upon a threshold for the initial amplitudes, for a generalised energy, whereas for an weighted energy, the nonlinear critical Rayleigh number guarantees *unconditional nonlinear stability.*

Concluding remarks

In the work presented in Part I we have seen how the energy method works for different fluids of grade n, precisely for $n = 1, 2, 3$. However, nothing is known regarding nonlinear stability results as one increases the grade n. For example, the stress tensor of the fourth grade fluid is of the form:

$$\begin{aligned}
\mathbf{T} = & - p\,\mathbf{I} + \mu\,\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\,\mathbf{A}_2)\,\mathbf{A}_1 \\
& + \gamma_1\,\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\
& + \gamma_5(tr\,\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(tr\,\mathbf{A}_2)\,\mathbf{A}_1^2 + [\gamma_7\,tr\mathbf{A}_3 + \gamma_8\,tr(\mathbf{A}_2\mathbf{A}_1)]\,\mathbf{A}_1.
\end{aligned}$$

To complete a nonlinear stability analysis for this fluid, one certainly needs specific information on μ, α_1, α_2, β_1, β_2, β_3, $\gamma_1, \gamma_2, \ldots \gamma_8$ - constants or temperature-dependent material moduli. Unfortunately, at the time of writing experimental work has not provided any helpful constraints on these material moduli. Also, a thermodynamic development of higher grade fluids is not available. Therefore, there are still questions to be answered regarding stability results for fluids of grade n, with $n > 3$.

On the other hand, work on viscoelastic fluids has employed the main assumption of a temperature-dependent viscosity, ignoring the temperature influence on the diffusivity or all other relevant physical quantities. Even if the present work looks restrictive from this point of view, it is useful to remark that for most fluids the dependence on temperature of the viscosity is much more significant than other physical quantities. Moreover, the results presented here may later help one to complete a more general nonlinear stability analysis of the Bénard problem for fluids of grade n when most of the physical quantities are temperature dependent.

Appendix A

Useful inequalities

Throughout we have refered to various inequalities which are defined below:

1. *Cauchy-Schwarz inequality.*

$$\langle a\, b\rangle \leq \left(\int_V a^2\, dV\right)^{1/2}\left(\int_V b^2\, dV\right)^{1/2} = ||a||\, ||b||, \tag{A.0.1}$$

where $a, b \in L^2(V)$, with all notations defined in the introductory part.

2. *Poincaré inequality.*

(i) For arbitrary functions s regular in the three-dimensional region Ω, vanishing on the boundary and such that $\nabla s \in L^2(\Omega)$, one has

$$\int_\Omega (\nabla s)^2\, d\Omega \geq \lambda_1 \int_\Omega s^2\, d\Omega, \tag{A.0.2}$$

the positive constant λ_1 depending on the geometric form of the domain Ω.

(ii) One also has

$$\int_0^1 (s_x)^2\, dx \geq \lambda_1 \int_0^1 s^2\, dx,$$

for arbitrary regular functions $s \in C(0,1)$ with $s_x \in L^2(0,1)$ and either $s(0) = 0$ or $s(1) = 0$.

3. *Sobolev inequality.*

For a regular function s on V (V defined in the introductory part of this thesis), which vanishes at $z = 0$,

$$||s^2||^2 \leq \gamma^4 \, ||\nabla s||^4, \tag{A.0.3}$$

with

$$\gamma^4 = \frac{16}{\sigma^2 \pi^3}(1 + M\pi + 1/4\,\pi^2\,(\sigma^2 + M^2))$$

where $M = \max_\Lambda |x_i|$ and $\sigma = \min_\Gamma x_i n_i$, Λ being the cross section of the periodic cell, with the boundary Γ_Λ. Here n denotes the unit outward normal vector to Γ_Λ (cf. Galdi et al. [26], p.101).

4. *A boundary estimation for s^2.*

For $s \in W^{1,2}(V)$ a function that vanishes on $z = 0$

$$\int_\Gamma s^2 \, dA \leq 2||s|| \, ||\nabla s||, \tag{A.0.4}$$

where Γ is the boundary of the periodic cell V, lying in the $z = 1$ plane.

To see that (A.0.4) holds for $s = 0$ at $z = 0$, note that

$$s^2(x, y, 1) = 2 \int_0^1 s \, \frac{\partial s}{\partial z} \, dz.$$

Integrating over V and using the Cauchy-Schwarz inequality, leads to (A.0.4) (cf. Galdi et al. [26], p.99).

The next three inequalities are strictly related to the analyses and notation of this thesis. They are called in the conditional nonlinear stability analysis for the thawing subsea permafrost model, Chapter 7, Subsection 7.3.1.

5. *Useful inequality for $||s^2||$.*

For an arbitrary function $s \in L^4(V)$, with $s = 0$ for $z = 0$, one may get

$$||s^2|| \leq \frac{E^{1/4}}{\mu^{1/4}}\sqrt{2}, \tag{A.0.5}$$

where the generalised energy function, $E(t)$, is defined by

$$E(t) = \frac{1}{2}||s||^2 + \frac{\mu}{4}||s^2||^2$$

with μ a positive parameter.

6. *Useful inequality for* $||w||^{1/2}$.

One has

$$||w||^{1/2} \leq \mathcal{D}^{1/4}, \tag{A.0.6}$$

where $w = u_3$ and $\mathcal{D} = ||\mathbf{u}||^2 + ||\nabla s||^2 + a \int_\Gamma s^2 \, dA$.

7. *Inequality for* $D^{1/2}(s)$.

One can obtain

$$D^{1/2}(s) \leq \mathcal{D}^{1/2}, \tag{A.0.7}$$

since $\mathcal{D} = ||u||^2 + D(s) + a \int_\Gamma s^2 \, dA$, with $D(s) = ||\nabla s||^2$ being the Dirichlet integral over V.

Appendix B

The compound matrix method

Before presenting the method used throughout for the numerical calculations, we stress that the number of compound matrix equations are ${}^{2n}C_n$ where $2n$ is the order of the original differential equation system. For thawing subsea permafrost the initial system is of order 4, whereas most of the systems to be solved in the stability study for viscoelastic fluids are of order 6. Rather than describe the method to solve the system (1.14) for a Navier-Stokes fluid (20 equations in the compound matrix), we prefer to select the equations (7.18) from Part B (only 6 equations in the compound matrix). For all the systems from the first part, the method presented below still works, though the computation is more laborious due to the need to calculate and then numerically integrate 20 equations.

Let us consider now the equations (7.18), i.e.

$$\begin{aligned}(D^2-k^2)W+R\,g_1(z)k^2S &= 0,\\ (D^2-k^2)S+RW &= \sigma S,\end{aligned}$$

with the boundary conditions given by

$$\begin{array}{lll}(BC) & W=0, & z=0,1\\ (BC)_0 & S=0, & z=0\\ (BC)_1 & DS+aS=0, & z=1.\end{array}$$

We now approximate the solution to the system above using the compound matrix method (cf. Straughan & Walker, [67]). To understand the necessity of using the compound matrix method we first employ the shooting method.

We introduce the vector

$$V = (W, W', S, S')^T \tag{B.0.1}$$

and replace the boundary conditions (7.19) at $z = 0$ on W, S by

$$W'(0) = 1, \quad S'(0) = 0 \quad \text{and} \quad W'(0) = 0, \quad S'(0) = 1. \tag{B.0.2}$$

Then let $V_1 = (W_1, W_1', S_1, S_1')^T$ and $V_2 = (W_2, W_2', S_2, S_2')^T$ be the independent solutions for the initial values $(W(0), W'(0), S(0), S'(0)) = (0, 1, 0, 0)$, respectively $(W(0), W'(0), S(0), S'(0)) = (0, 0, 0, 1)$. It follows that the final solution, V, is a linear combination of these two solutions

$$V = \alpha V_1 + \beta V_2 = \begin{pmatrix} \alpha W_1 + \beta W_2 \\ \alpha W_1' + \beta W_2' \\ \alpha S_1 + \beta S_2 \\ \alpha S_1' + \beta S_2' \end{pmatrix} = \begin{pmatrix} W \\ W' \\ S \\ S' \end{pmatrix}.$$

We impose now the correct boundary condition at $z = 1$ and this requires

$$\begin{aligned} &W(1) = 0, \\ &S'(1) + aS(1) = 0, \end{aligned}$$

or in matrix formulation

$$\begin{pmatrix} W_1(1) & W_2(1) \\ S_1'(1) + aS_1(1) & S_2'(1) + aS_2(1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We require a non-zero solution (α, β). Therefore, the determinant of the matrix has to be zero. Thus the condition at $z = 1$ is finally

$$W_1(1)[S_2'(1) + aS_2(1)] - W_2(1)[S_1'(1) + aS_1(1)] = 0, \tag{B.0.3}$$

which may lead to numerical instability due to round off errors because we may be subtracting very large and nearly equal quantities.

One way to avoid this is to employ the compound matrix technique. The idea is to remove the troublesome location of the zero of a determinant by converting to a system of ordinary differential equations in the determinants themselves.

The compound matrix method introduces new y_i (i=1,2,3,4,5,6) variables and works directly with these. The six vector $Y = (y_1, y_2, y_3, y_4, y_5, y_6)^T$ is formed with the 2×2 minors of the matrix with columns V_1 and V_2

$$\begin{pmatrix} W_1 & W_2 \\ W_1' & W_2' \\ S_1 & S_2 \\ S_1' & S_2' \end{pmatrix}$$

and the new six variables are

$$\begin{aligned}
y_1 &= W_1 W_2' - W_2 W_1', \\
y_2 &= W_1 S_2 - W_2 S_1, \\
y_3 &= W_1 S_2' - W_2 S_1', \\
y_4 &= W_1' S_2 - W_2' S_1, \\
y_5 &= W_1' S_2' - W_2' S_1', \\
y_6 &= S_1 S_2' - S_2 S_1'.
\end{aligned}$$

In this new environment the correct condition at $z = 1$ is (B.0.3) and this becomes

$$y_3(1) + a y_2(1) = 0, \qquad \text{(final condition)} . \tag{B.0.4}$$

The y_i 's satisfy the system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \tag{B.0.5}$$

where A

$$\mathbf{A} = \begin{pmatrix} 0 & -R\,g_1(z)\,k^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & k^2 & 0 & 0 & 1 & 0 \\ 0 & k^2 & 0 & 0 & 1 & 0 \\ R & 0 & k^2 & k^2 & 0 & -R\,g_1(z)\,k^2 \\ 0 & R & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues may be found by integrating (B.0.5) from 0 to 1 with the initial condition

$$y_5(0) = 1, \tag{B.0.6}$$

and iterating on the final condition

$$y_3(1) + ay_2(1) = 0. \tag{B.0.7}$$

In order to find a linear instability threshold we take $\sigma = 0$ in (7.18) and find critical values of R and the wavenumber k. A priori we cannot assume $\sigma \in \mathbb{R}$, which is equivalent to $\sigma = 0$. However, we note that Hutter & Straughan [38] find $\sigma \in \mathbb{R}$ in their analysis and so we expect it to be true here. Since the unconditional nonlinear stability result is found to be close to the linear one this provides further justification for this procedure.

The first step is to vary R until the final condition is satisfied to some pre-assigned tolerance and find that particular value of R with the secant method. We then find numerically

$$Ra_L = R_L^2 = \min_{k^2} R^2(k^2, a),$$

varying k^2, by using the golden section search algorithm. The Rayleigh number and the wave number are dependent on a, so we run the same routine for different values of $a > 0$.

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